Modelling of Ridge Waveguide Bends for Sensor Applications

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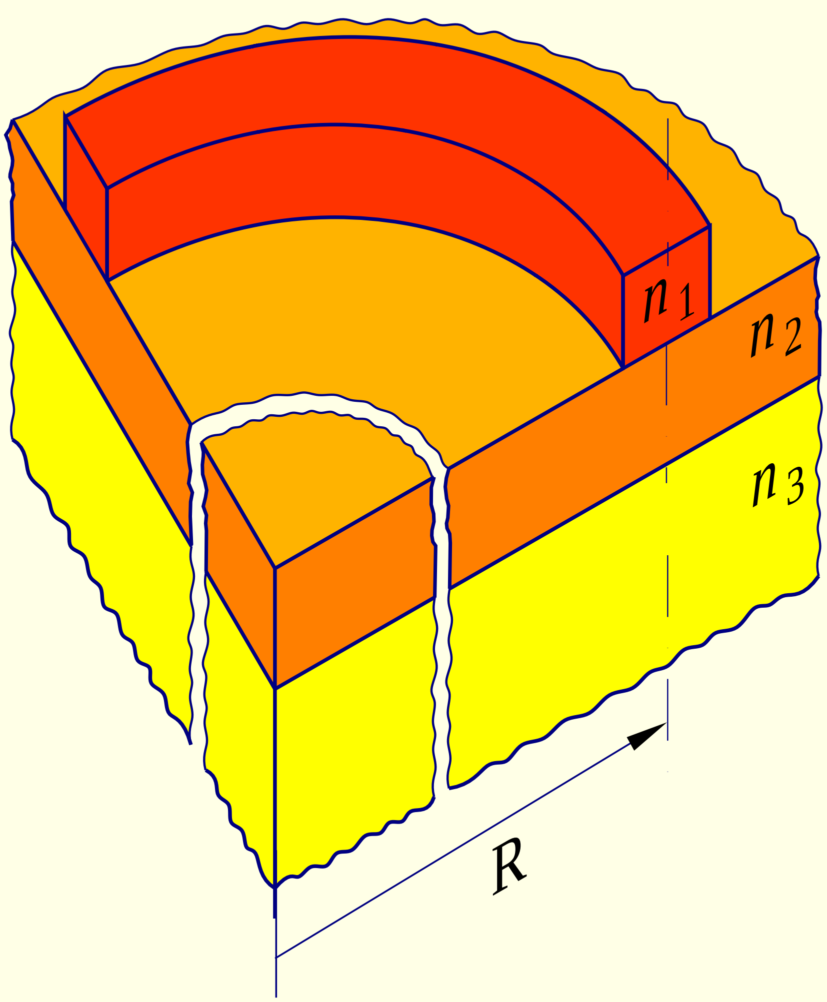
FernUniversität, Hagen, Germany



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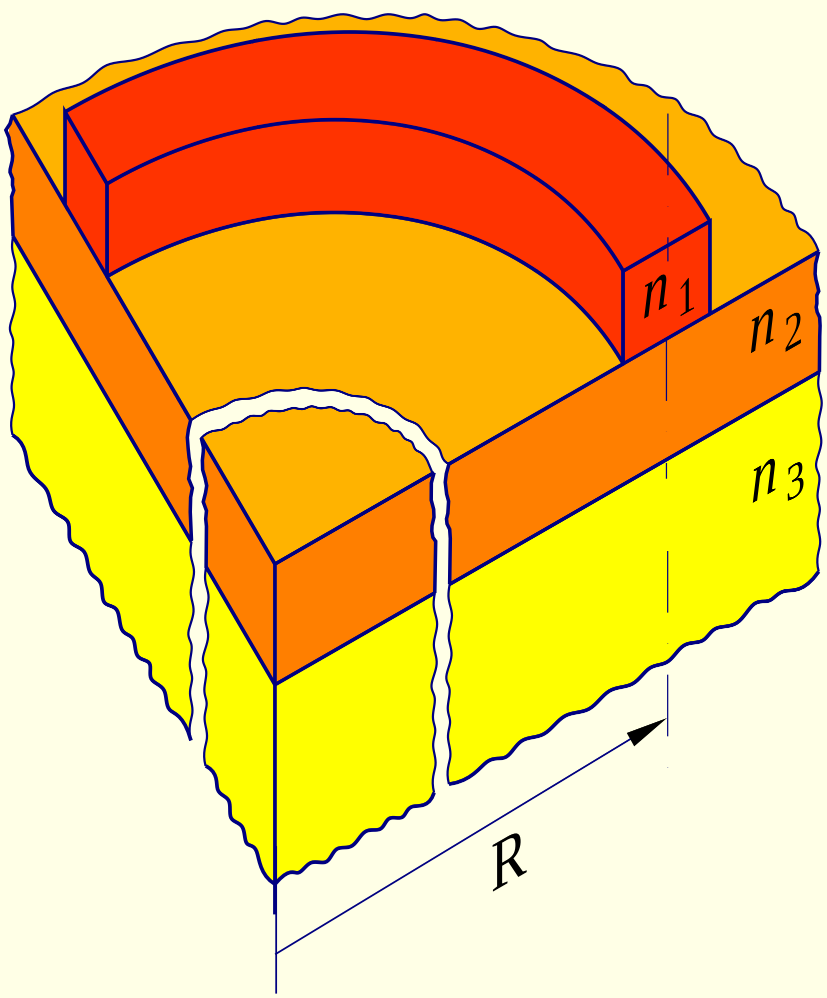

- Radiation losses
- Evanescent field of a rib waveguide
- are precisely modelled by the Method of Lines

Why employ the Method of Lines?

The MoL is a semianalytic approach

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  (with Finite Differences)
  $\Longrightarrow$ 3D problem $\longrightarrow$ 2D discretization

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For reasons of technology, waveguide structures are

- multilayered (e.g., planar waveguides)
- cascaded (e.g., waveguide circuits)

Advantages and Disadvantages

+ precise modeling
+ low memory and computing time

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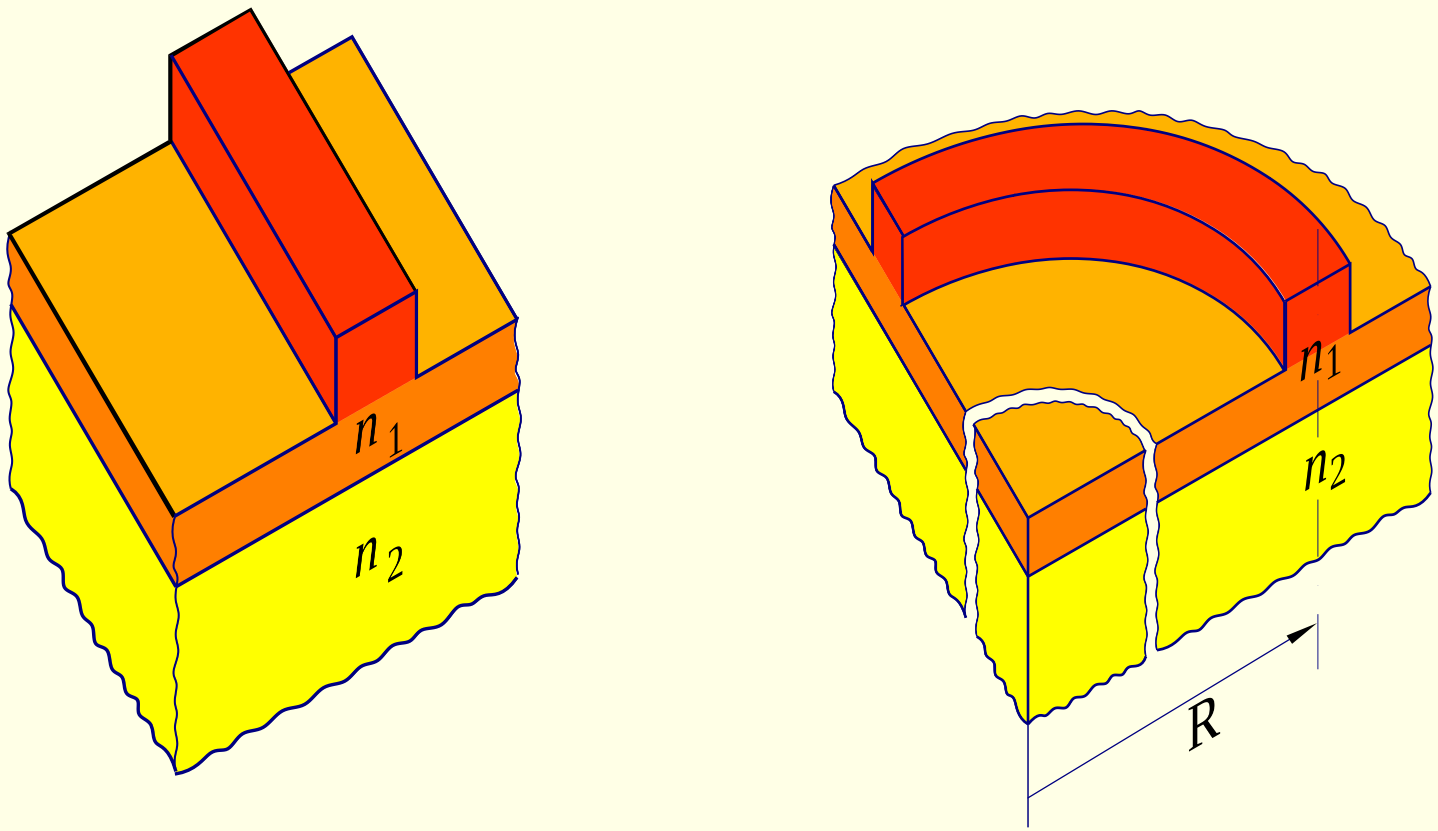
+ precise modeling
+ low memory and computing time
– reduced flexibility
  - $\Longrightarrow$ different geometries require new algorithms
  - $\Longrightarrow$ extension to hybrid methods

Transition to Cylindrical Coordinates



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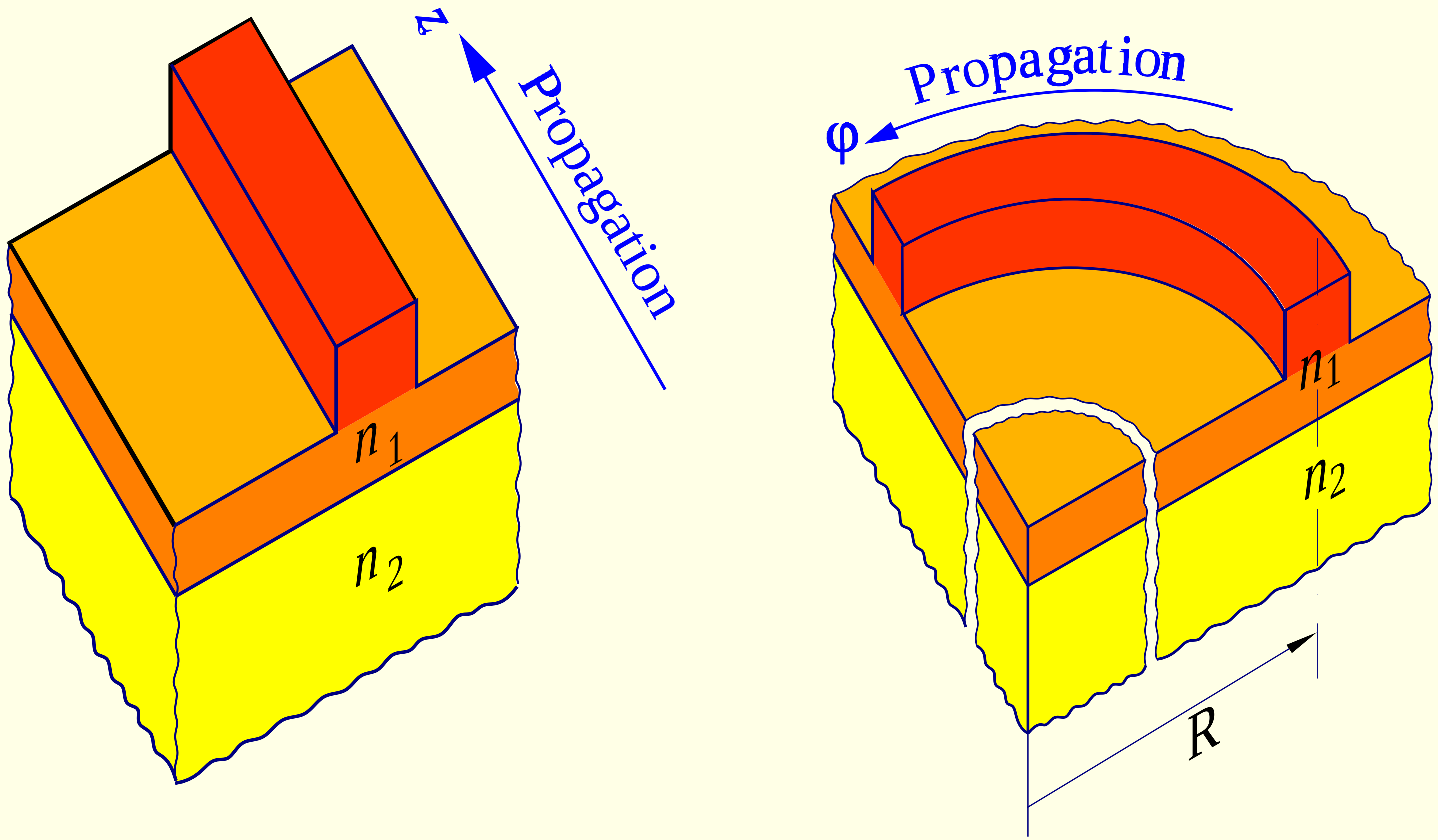

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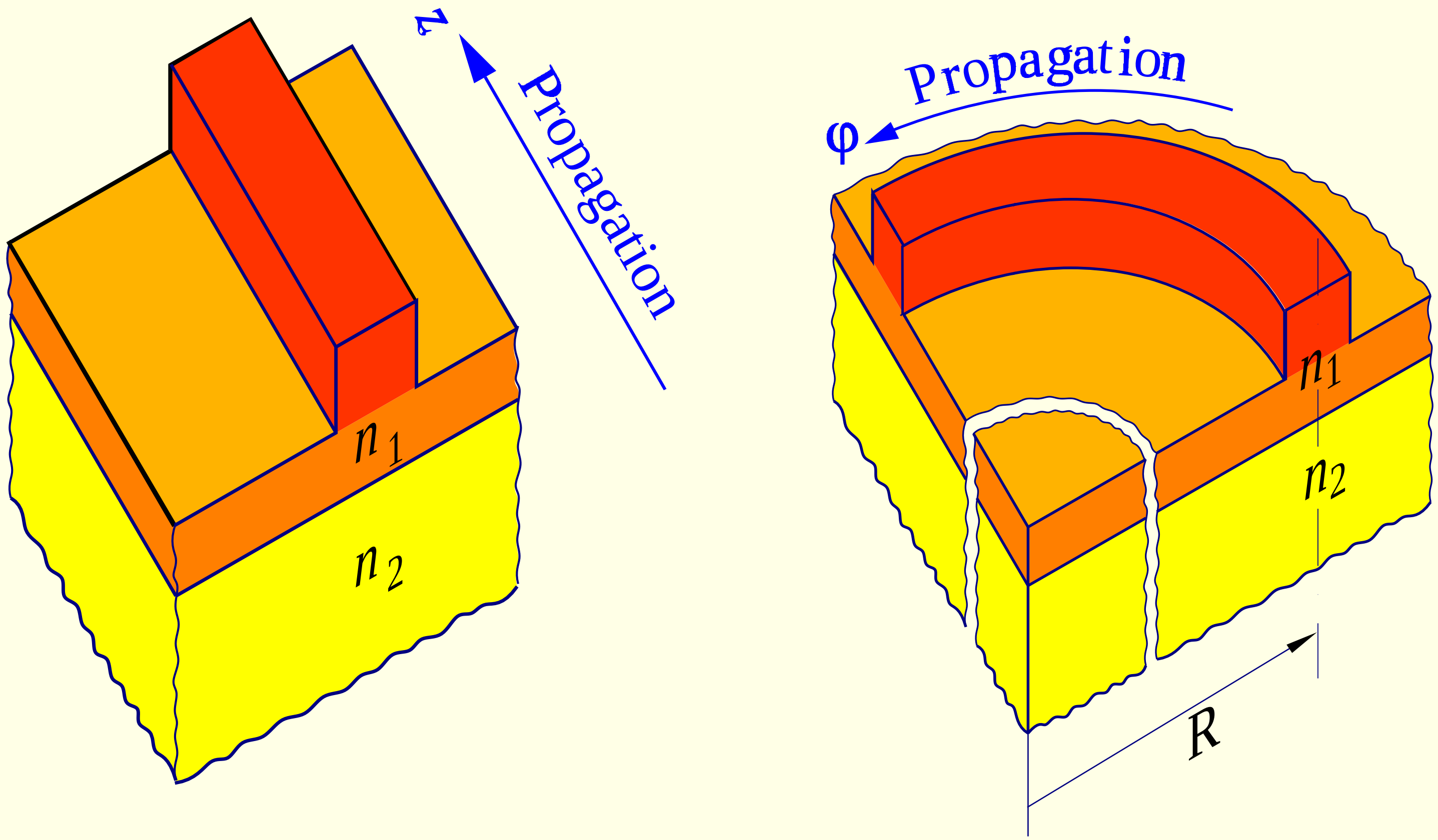

$$\text{Propagation} \quad z \to \varphi$$

$$\exp(-j\beta z) \to \exp(-j\nu\varphi)$$

Discretization of Straight / Bent Waveguides

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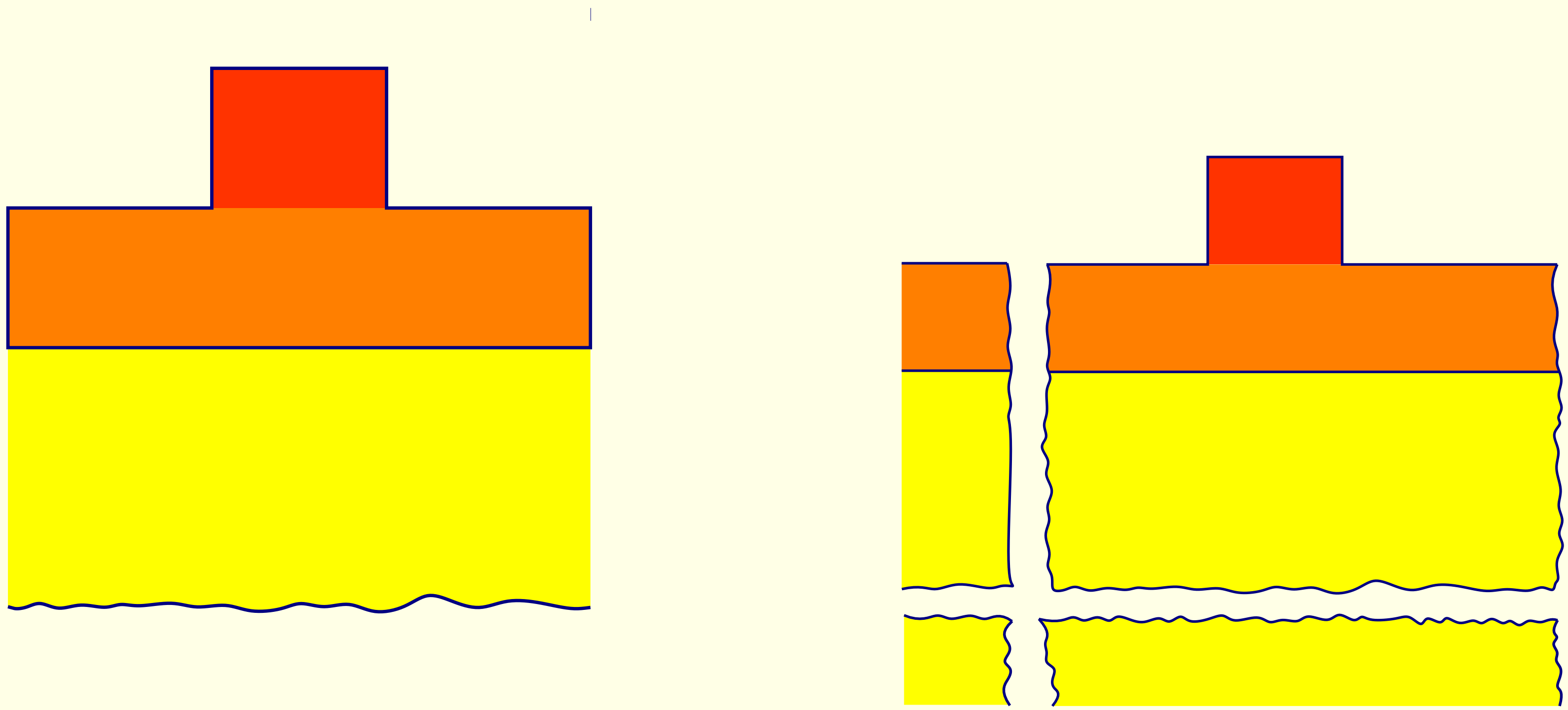

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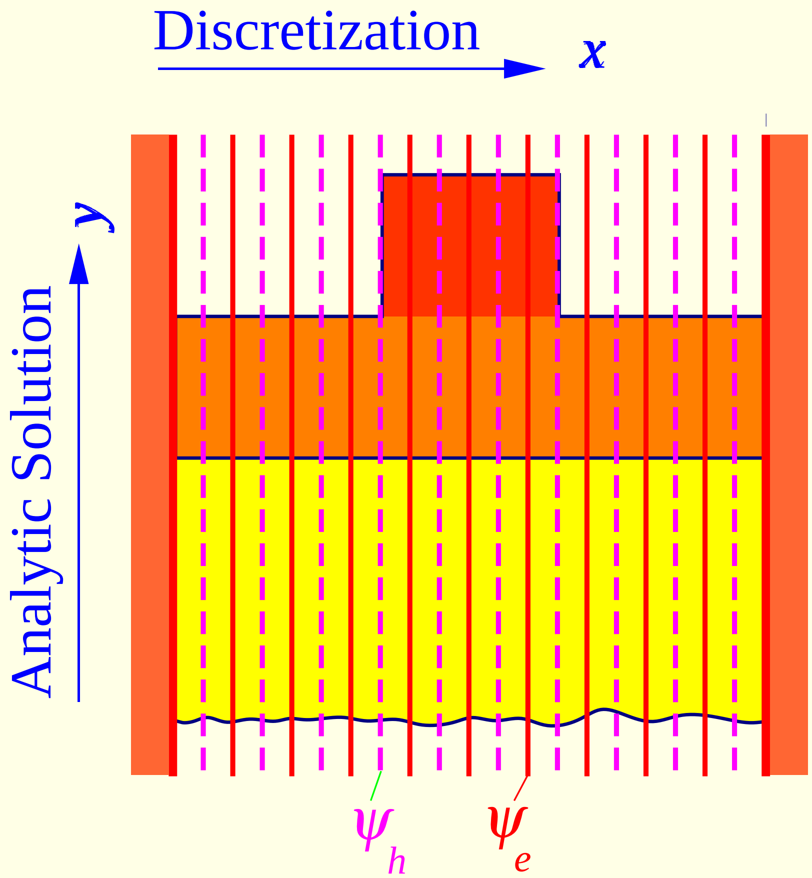

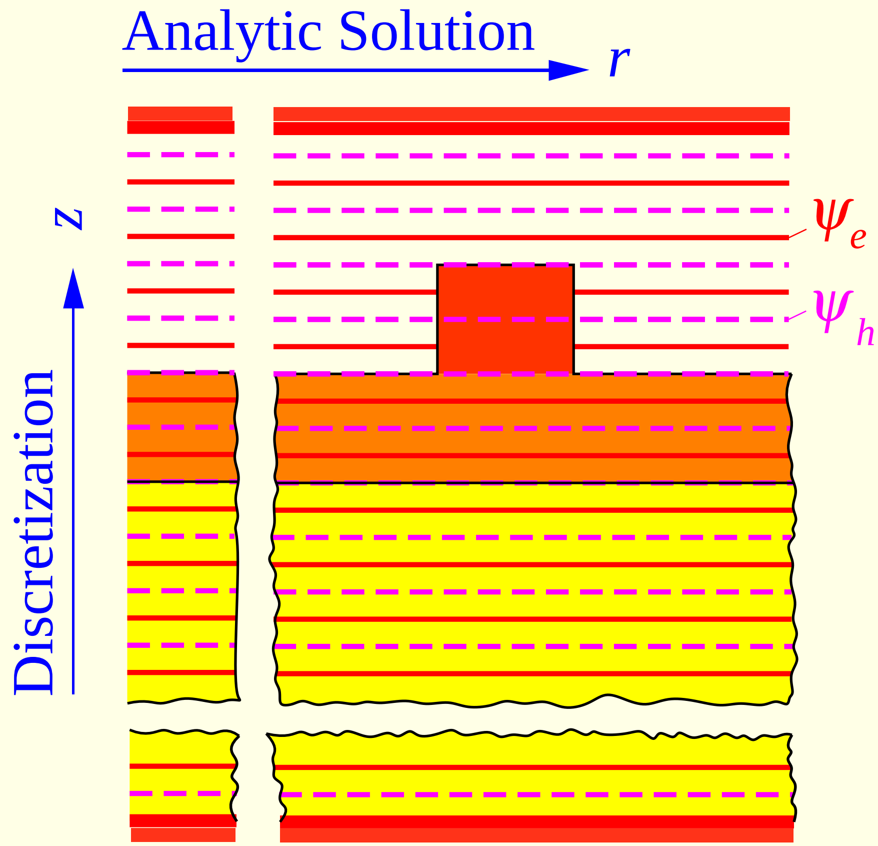

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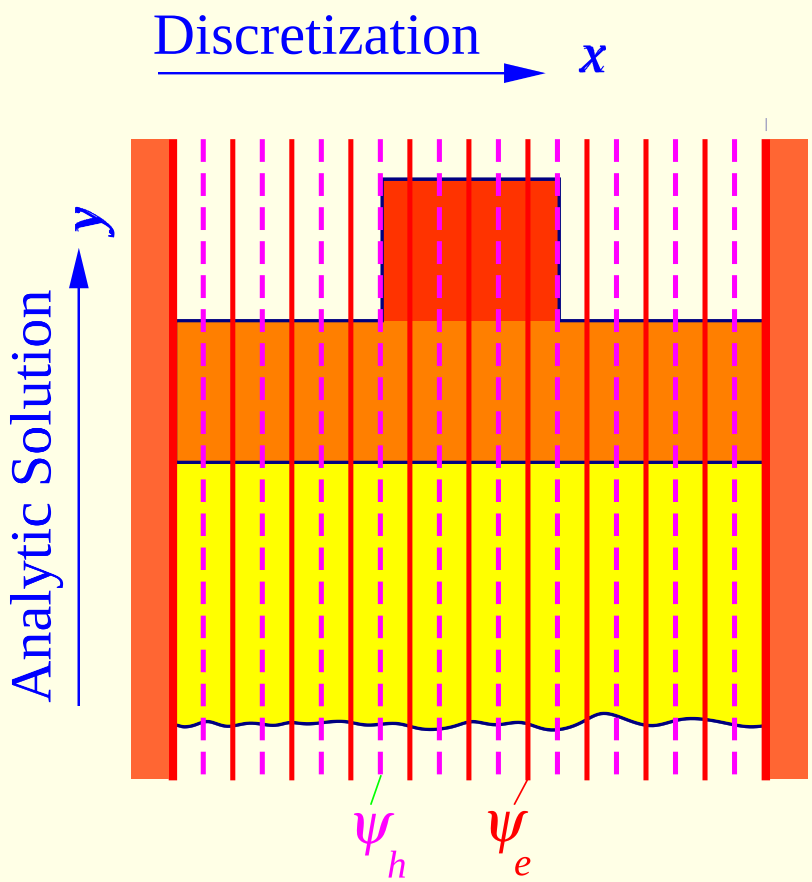

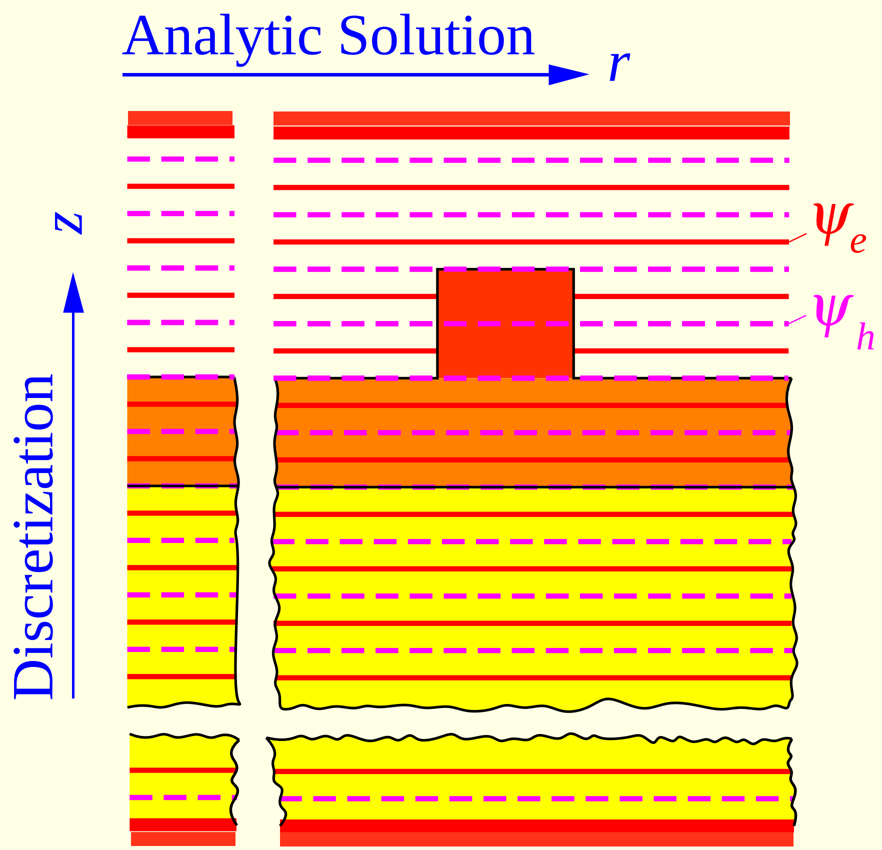

Discretization $x \rightarrow z$

$$\boldsymbol{P}_x \rightarrow \boldsymbol{P}_z$$

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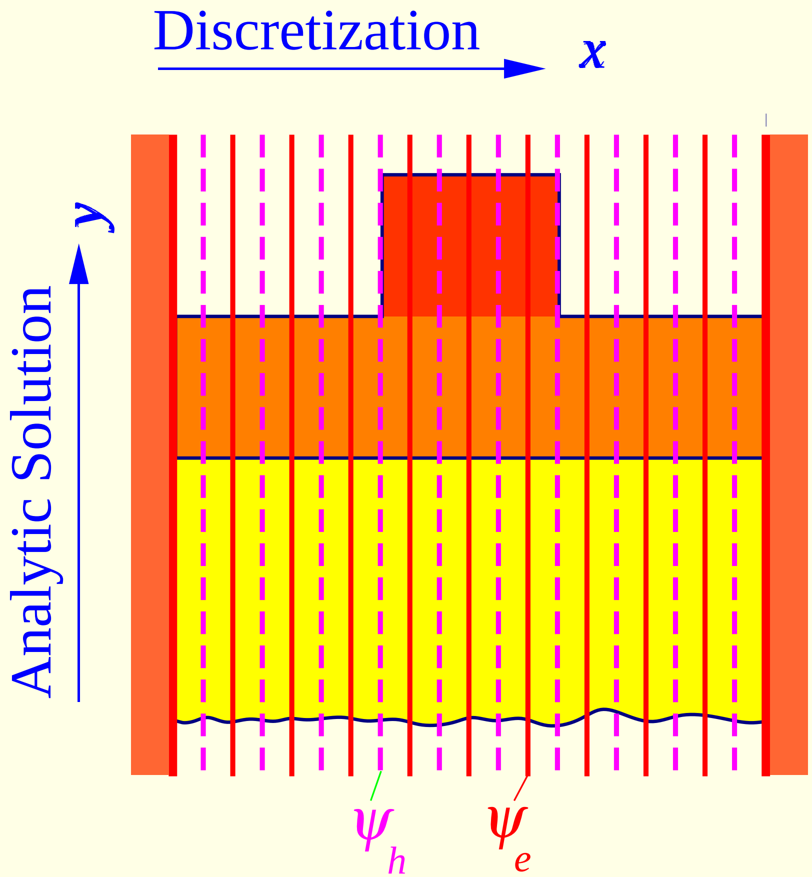

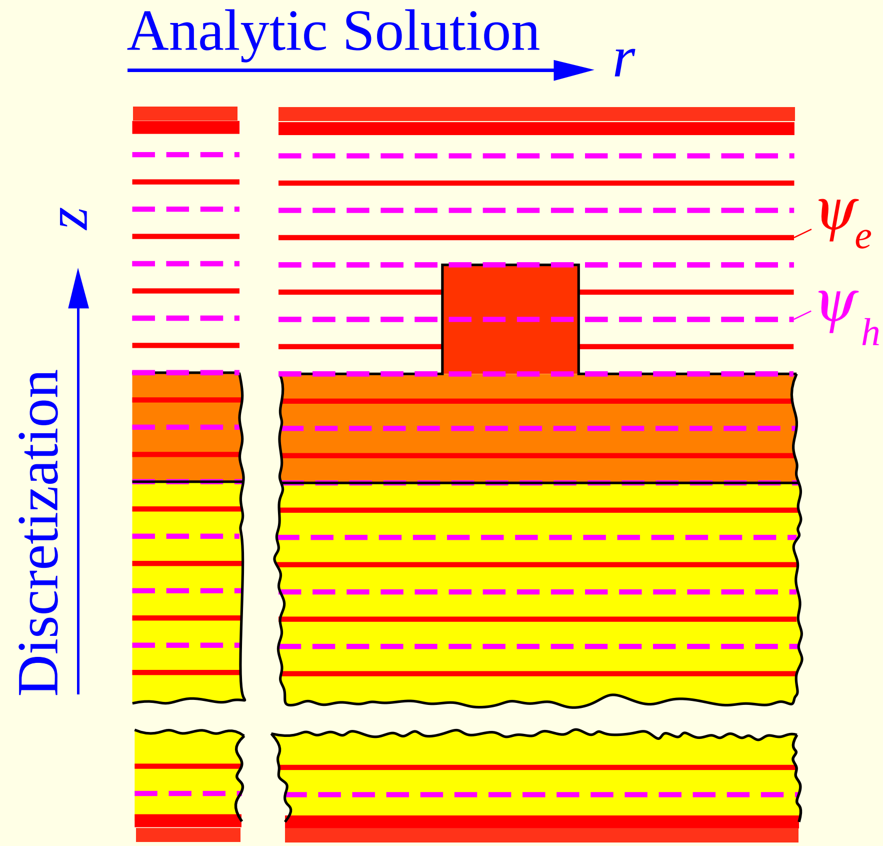

Discretization $x \to z$

$$\boldsymbol{P}_x \to \boldsymbol{P}_z$$

Analytic solution $y \to r$

$$\sin(k_{\overline{y}}\overline{y}) \to J_\nu(\widetilde{\lambda}\overline{r})$$

The Method of Lines (MoL)
for circular bends in waveguides

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7. Characteristic equation in C

   $\longrightarrow$ radiation loss $\; L \propto \mathrm{Im}(n_{eff})$

Wave equations in coordinate free form

Vector MoL with two potentials $\mathbf{\Pi}_e$, $\mathbf{\Pi}_h$

$+++$ accurate fulfillment of the continuity conditions for all field components

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$\longrightarrow$ *coordinate free* approach:

a) Helmholtz equation for $\mathbf{\Pi}_h$

$$\left\{ \Delta \, + \, \varepsilon_r(z)k_0^2 \right\} \mathbf{\Pi}_h = \mathbf{0}$$

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b) Sturm-Liouville differential equation for $\mathbf{\Pi}_e$

$$\left\{ \Delta + \varepsilon_r(z) k_0^2 - \frac{1}{\varepsilon_r(z)} \operatorname{grad} \varepsilon_r(z) \cdot \operatorname{div} \right\} \mathbf{\Pi}_e = \mathbf{0}$$

Wave equations in *cylindrical coordinates*

Potentials with **one** component in z direction only

$$\mathbf{\Pi}_{e,h} = k_0^{-2} \ \exp(-j\nu\varphi) \ \psi_{e,h} \ \mathbf{a}_z$$

- order proportional to effective model index $\nu \quad = \quad n_{eff}\overline{R}$

using normalized coordinates: e.g. $\overline{R} \quad = \quad k_0 R$

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Consideration of the radiation losses

$\Rightarrow$ n_{eff}, ν *complex*

$+ + +$ no artificial increase in the guiding

Discretization of the wave equation in the cartesian z direction

Partial differential equations in cylindrical coordinates

$$\left\{ \frac{1}{\overline{r}} \frac{\partial}{\partial \overline{r}} \left(\overline{r} \frac{\partial}{\partial \overline{r}} \right) - \frac{\nu^2}{\overline{r}^2} + \varepsilon_r(z) \quad + \quad \frac{\partial^2}{\partial \overline{z}^2} \right\} \psi_h = 0$$

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Potentials and dielectric constants

continuous	$\longrightarrow$	**discretized**	
$\psi_e \, , \psi_h$	$\longrightarrow$	$\mathbf{\Psi}_e \, , \mathbf{\Psi}_h$	(column vector)
$\varepsilon_r(z)$	$\longrightarrow$	$\boldsymbol{\varepsilon}_e \, , \boldsymbol{\varepsilon}_h$	(diagonal matrix)

Discretization of the wave equation in the cartesian z direction

Differential operators $\longrightarrow$ **difference operators**

$$\frac{\partial^2}{\partial \overline{z}^2}\ \psi_h \quad \longrightarrow \quad -\boldsymbol{P}_{zh}\ \boldsymbol{\Psi}_h \qquad \text{(tridiagonal)}$$

$$\varepsilon_r(z)\frac{\partial}{\partial \overline{z}}\left(\frac{1}{\varepsilon_r(z)}\frac{\partial}{\partial \overline{z}}\right)\ \psi_e \quad \longrightarrow \quad -\boldsymbol{P}^{\varepsilon}_{ze}\ \boldsymbol{\Psi}_e \qquad \text{(tridiagonal)}$$

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$\longrightarrow$ Coupled ordinary differential equations

$$\left\{ \frac{1}{\overline{r}}\frac{d}{d\overline{r}}\left(\overline{r}\frac{d}{d\overline{r}}\right)\boldsymbol{I} - \frac{\nu^2}{\overline{r}^2}\boldsymbol{I} + \underbrace{\boldsymbol{\varepsilon}_{e,h} - \left\{ \begin{matrix} \boldsymbol{P}^{\varepsilon}_{ze} \\ \boldsymbol{P}_{zh} \end{matrix} \right\}}_{\text{(tridiagonal)}} \right\} \boldsymbol{\Psi}_{e,h} = \boldsymbol{0}$$

Transformation to diagonal form

with $$\boldsymbol{T}_e^{\varepsilon\,-1}\;\overbrace{(\boldsymbol{\varepsilon}_e - \boldsymbol{P}_{ze}^{\varepsilon})}^{\text{(tridiagonal)}}\;\boldsymbol{T}_e^{\varepsilon} \;=\; \widetilde{\boldsymbol{\lambda}}_e^2 \;=\; \operatorname{diag}\left(\widetilde{\boldsymbol{\lambda}}_{e,k}^2\right)$$

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with the transformed potential

$$\widetilde{\boldsymbol{\Psi}}_e = \boldsymbol{T}_e^{\varepsilon\,-1} \boldsymbol{\Psi}_e$$

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$$\text{with} \qquad \boldsymbol{T}_e^{\varepsilon\,-1} \overbrace{(\boldsymbol{\varepsilon}_e - \boldsymbol{P}_{ze}^{\varepsilon})}^{\text{(tridiagonal)}} \boldsymbol{T}_e^{\varepsilon} \;=\; \widetilde{\boldsymbol{\lambda}}_e^2 \;=\; \operatorname{diag}(\widetilde{\boldsymbol{\lambda}}_{e,k}^2)$$

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$$\widetilde{\boldsymbol{\Psi}}_e = \boldsymbol{T}_e^{\varepsilon\,-1} \boldsymbol{\Psi}_e$$

completely analogous for $\widetilde{\boldsymbol{\Psi}}_h = \boldsymbol{T}_h^{\,-1} \boldsymbol{\Psi}_h$

Solution of the Bessel differential equation

$$\left\{ \frac{1}{\overline{r}} \frac{d}{d\overline{r}} \left(\overline{r} \frac{d}{d\overline{r}} \right) + \left(\widetilde{\lambda}^2_{e,k} - \frac{\nu^2}{\overline{r}^2} \right) \right\} \widetilde{\Psi}_{e,k} = 0 \quad \text{with} \quad \nu = n_{eff} \overline{R}_b$$

for **one** component $\widetilde{\Psi}_{e,k}$ of the transformed potential

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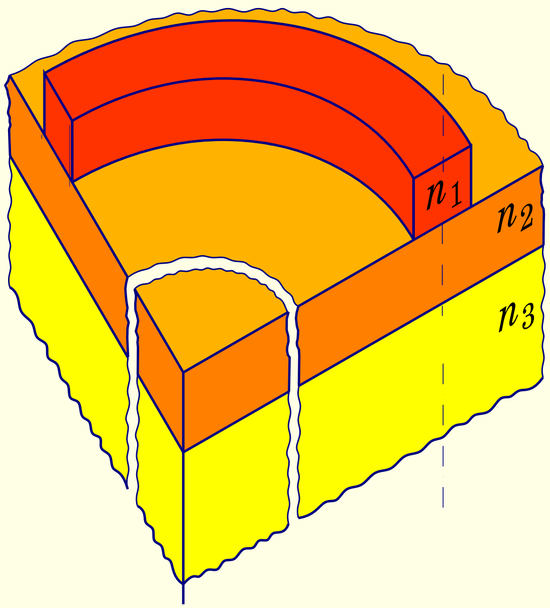

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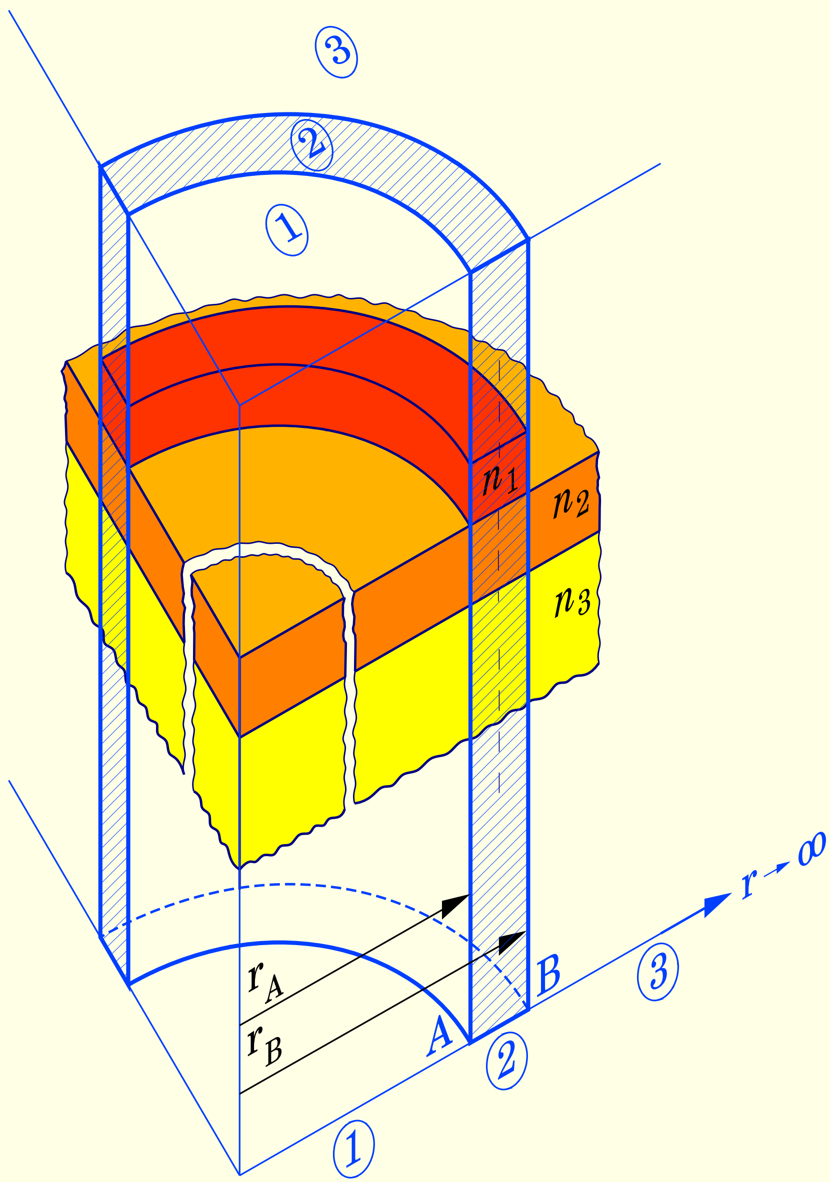
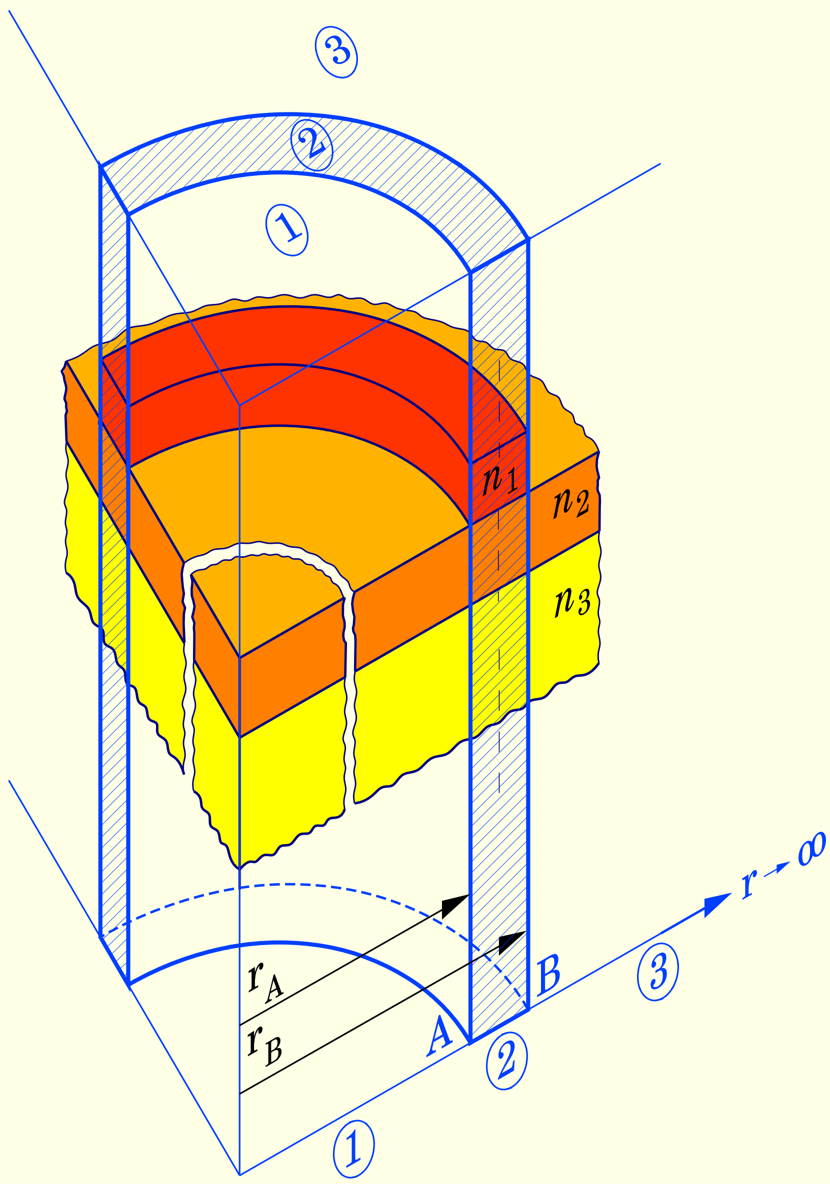


Solution in three regions

Solution of the Bessel differential equation



Solution in three regions

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$$\widetilde{\Psi}_{e,k} \;=\; A_k \; J_\nu(\widetilde{\lambda}_{e,k}\overline{r})$$

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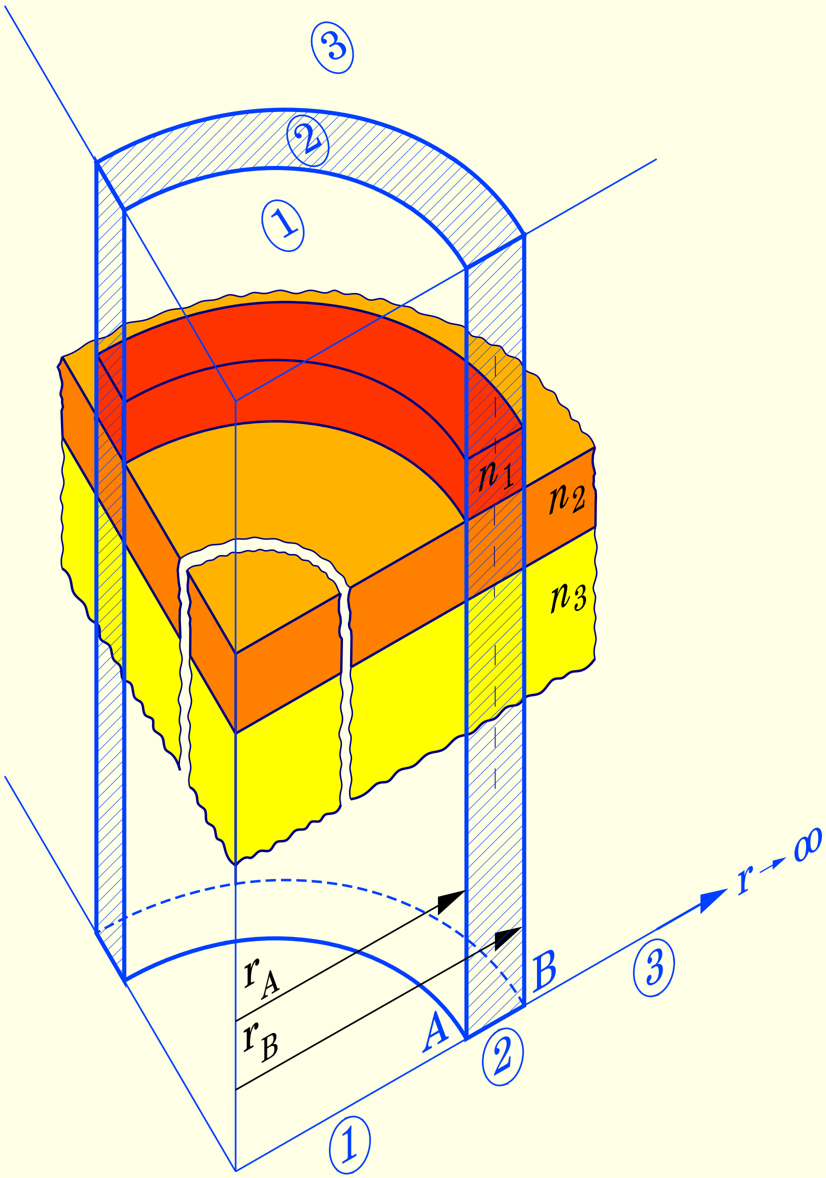

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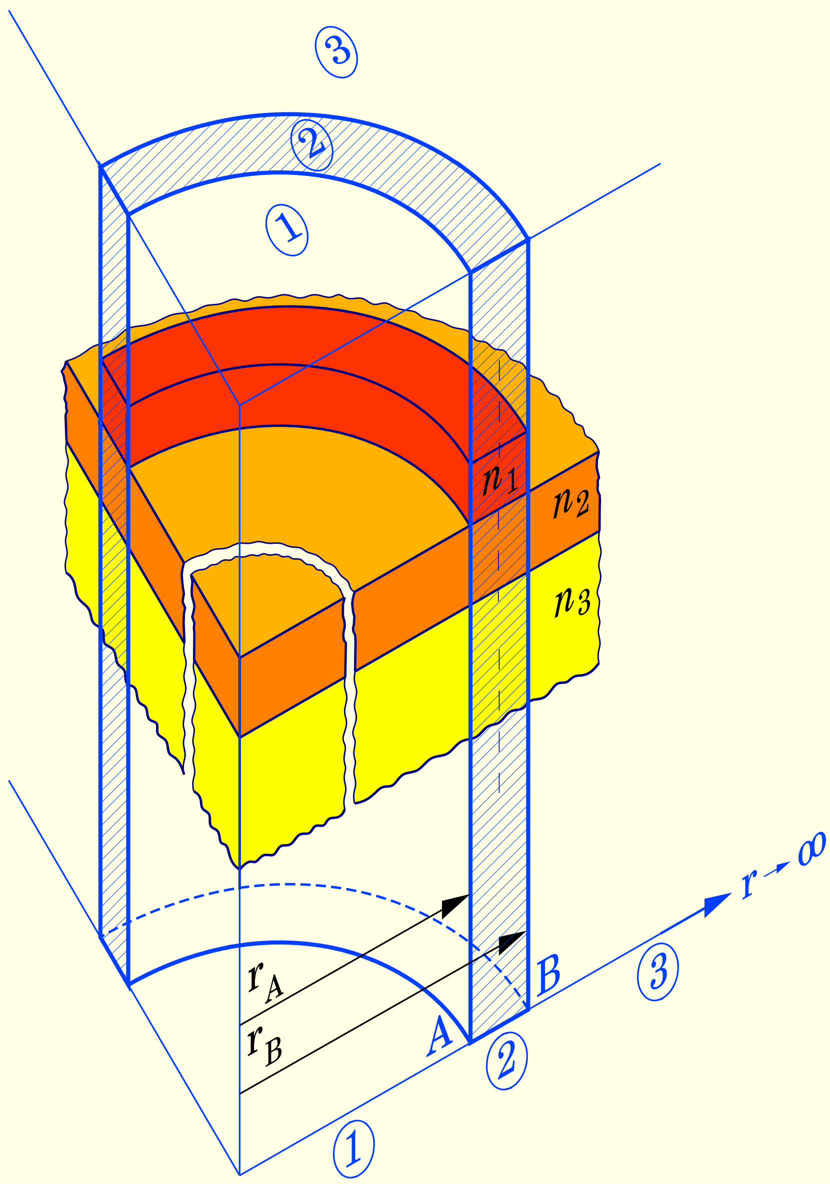

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3 radiation condition

$$\widetilde{\Psi}_{e,k} = D_k \; H_\nu^{(2)}(\widetilde{\lambda}_{e,k}\overline{r})$$

Radial derivation matrices $\boldsymbol{\Gamma}$

Transmission line equation

$$\frac{\partial}{\partial \overline{r}} \begin{bmatrix} \widetilde{\boldsymbol{\psi}}_A \\ \widetilde{\boldsymbol{\psi}}_B \end{bmatrix} = \boldsymbol{\Gamma} \begin{bmatrix} \widetilde{\boldsymbol{\psi}}_A \\ \widetilde{\boldsymbol{\psi}}_B \end{bmatrix} \text{ with } \quad \boldsymbol{\Gamma} = \frac{j\widetilde{\boldsymbol{\lambda}}}{\boldsymbol{p}_\nu} \begin{bmatrix} \boldsymbol{r}_\nu & \boldsymbol{W}_A \\ -\boldsymbol{W}_B & \boldsymbol{q}_\nu \end{bmatrix}$$

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The radial derivation matrix $\boldsymbol{\Gamma}$ is computed from the cross products and the Wronskian of the Bessel functions

$$\begin{aligned} \boldsymbol{p}_\nu &= J_\nu(\widetilde{\boldsymbol{r}}_A)Y_\nu(\widetilde{\boldsymbol{r}}_B) - J_\nu(\widetilde{\boldsymbol{r}}_B)Y_\nu(\widetilde{\boldsymbol{r}}_A) \\ \boldsymbol{q}_\nu &= J_\nu(\widetilde{\boldsymbol{r}}_A)Y'_\nu(\widetilde{\boldsymbol{r}}_B) - J'_\nu(\widetilde{\boldsymbol{r}}_B)Y_\nu(\widetilde{\boldsymbol{r}}_A) \\ \boldsymbol{r}_\nu &= J'_\nu(\widetilde{\boldsymbol{r}}_A)Y_\nu(\widetilde{\boldsymbol{r}}_B) - J_\nu(\widetilde{\boldsymbol{r}}_B)Y'_\nu(\widetilde{\boldsymbol{r}}_A) \\ \boldsymbol{W}_{A,B} &= \frac{2}{\pi \widetilde{\boldsymbol{r}}_{A,B}} \quad \text{with} \quad \widetilde{\boldsymbol{r}}_{A,B} = j\widetilde{\boldsymbol{\lambda}}\overline{r}_{A,B} \end{aligned}$$

Programming of the cylinder functions

The problem

1. very high complex order: $\nu \approx 10^5 - 5j$
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1, 2 $\Rightarrow$ Uniform asymptotic series (high argument and order)

Alternative for the ring regions

$\Rightarrow$ Multiplication formulas for cross products

Uniform asymptotic expansions

Bessel functions, e.g. Abramowitz-Stegun (9.3.35)

$$J_\nu(\nu y) \sim \left(\frac{4\zeta}{1-y^2}\right)^{1/4} \left\{ \frac{\mathrm{Ai}(\nu^{2/3}\zeta)}{\nu^{1/3}} \sum_{i=0}^{\infty} \frac{a_i(\zeta)}{\nu^{2i}} + \frac{\mathrm{Ai}'(\nu^{2/3}\zeta)}{\nu^{5/3}} \sum_{i=0}^{\infty} \frac{b_i(\zeta)}{\nu^{2i}} \right\}$$

$\nu \to \infty$

with $\quad \frac{2}{3}\zeta^{3/2} = \log \frac{1+\sqrt{1-y^2}}{y} - \sqrt{1-y^2}$

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$+++$ Series can be truncated already after $i = 2$

for double precision arithmetic

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$Y_\nu(\nu y)$, $J'_\nu(\nu y)$, $Y'_\nu(\nu y)$ computed analogously

Multiplication theorem
for direct calculation of cross products

Bessel function of outer radius $\widetilde{r}_B = \mu \widetilde{r}_A$

$$\mathcal{C}_\nu(\widetilde{r}_B) = \mathcal{C}_\nu(\mu \widetilde{r}_A) = \mu^{-\nu} \sum_{i=0}^{\infty} \frac{(\delta \widetilde{r}_A)^i}{i!} \quad \mathcal{C}_{\nu-i}(\widetilde{r}_A)$$

with $\delta = (\mu^2 - 1)/2$ Abramowitz-Stegun (9.1.74)

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yields series for cross products p_ν, q_ν

$$\left\{ \begin{array}{c} p_\nu \\ q_\nu \end{array} \right\} = \mu^{-\nu} \sum_{i=0}^{\infty} \frac{(\delta \widetilde{r}_A)^i}{i!} \left\{ \begin{array}{c} P_{\nu,i}(\widetilde{r}_A) \\ Q_{\nu,i}(\widetilde{r}_A) \end{array} \right\}$$

Multiplication theorem
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$\Longrightarrow$ Two independent procedures for checking

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- in every subregion separately
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$\rightarrow$ now $\widetilde{\boldsymbol{y}}_{1,2}$ depend on the radii r_A, r_B
$\rightarrow$ now $\widetilde{\boldsymbol{y}}_{1,2}$ are computed using cylinder functions

Determination of the radiation losses

Inverse transformation to spatial domain

$\longrightarrow$ characteristic equation

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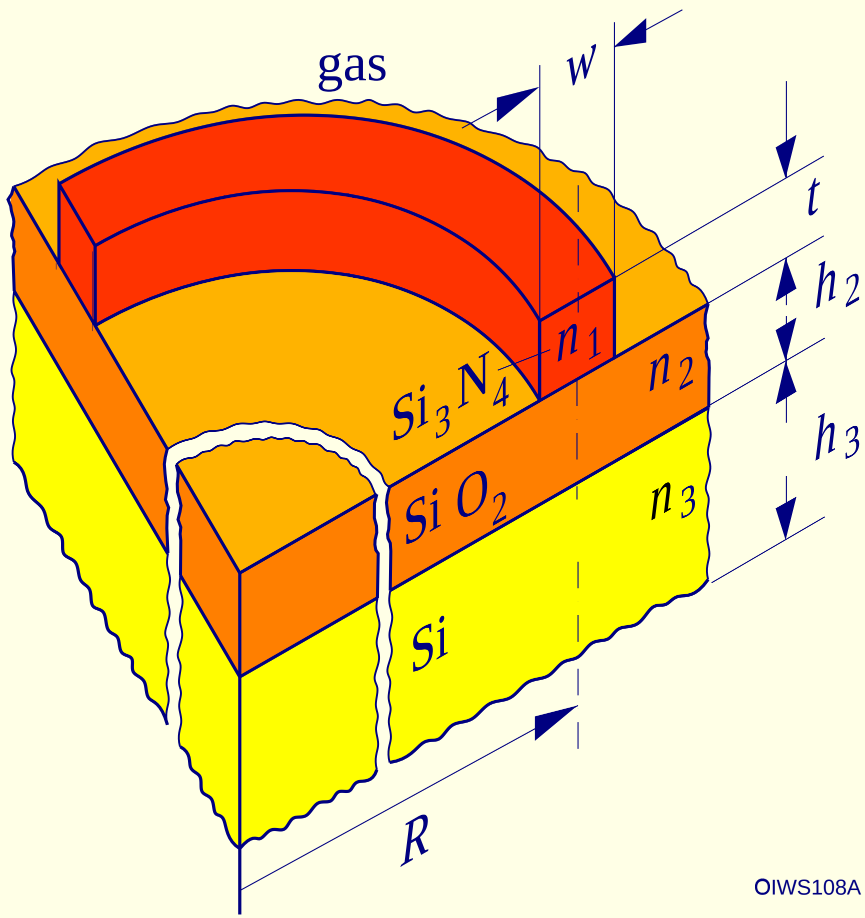
$\longrightarrow$ **Radiation losses** (dB/90°):

$$L = \mathrm{Im}(n_{eff})\ \cdot\ \overline{R}\ \pi \frac{10}{\ln 10}$$

Geometry of a Bent Rib Waveguide Sensor



OIWS108A

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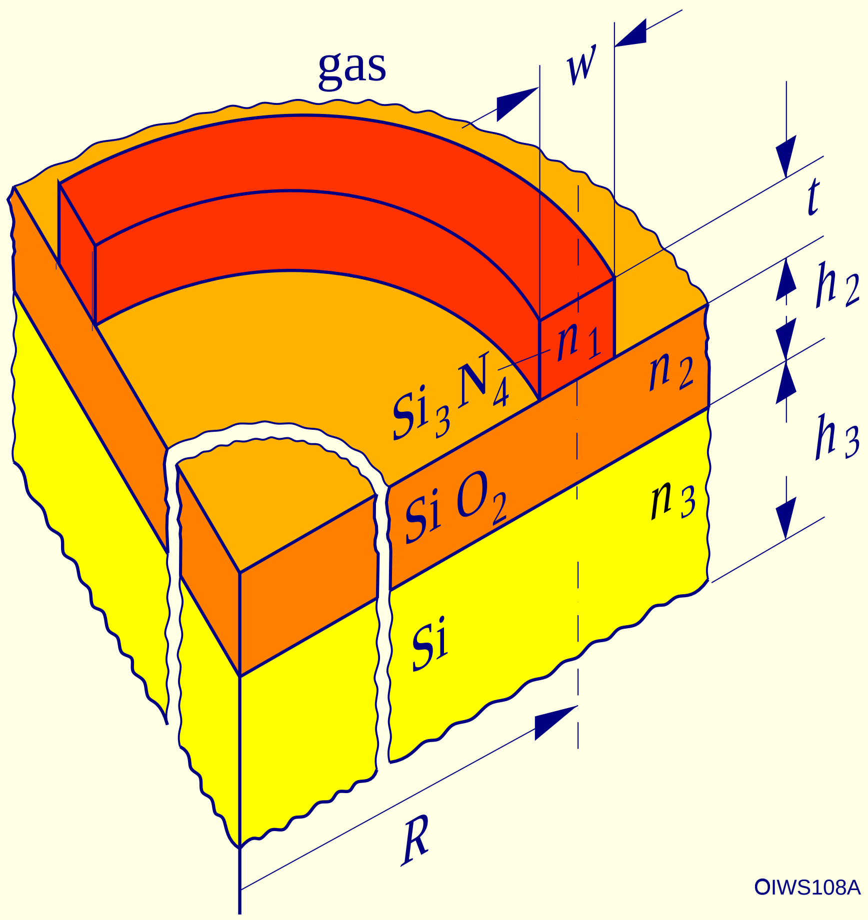
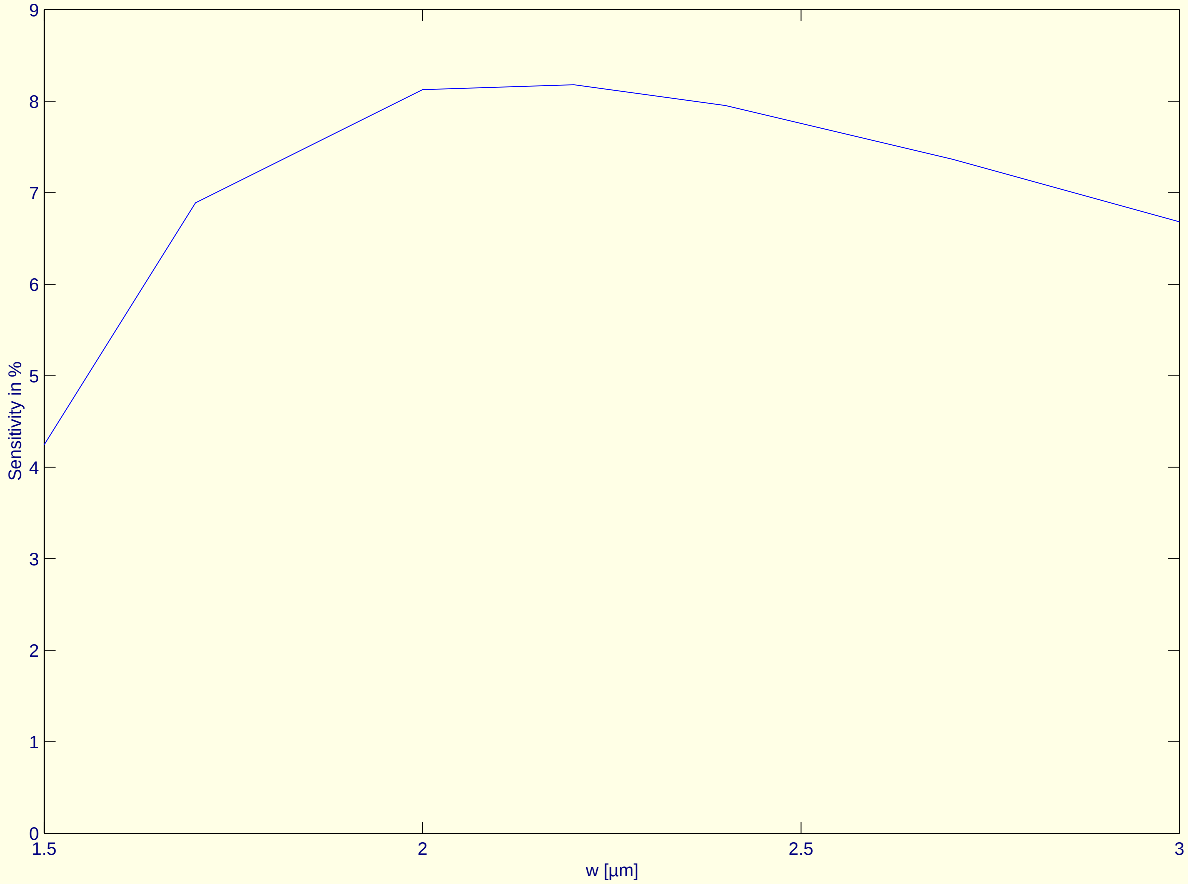


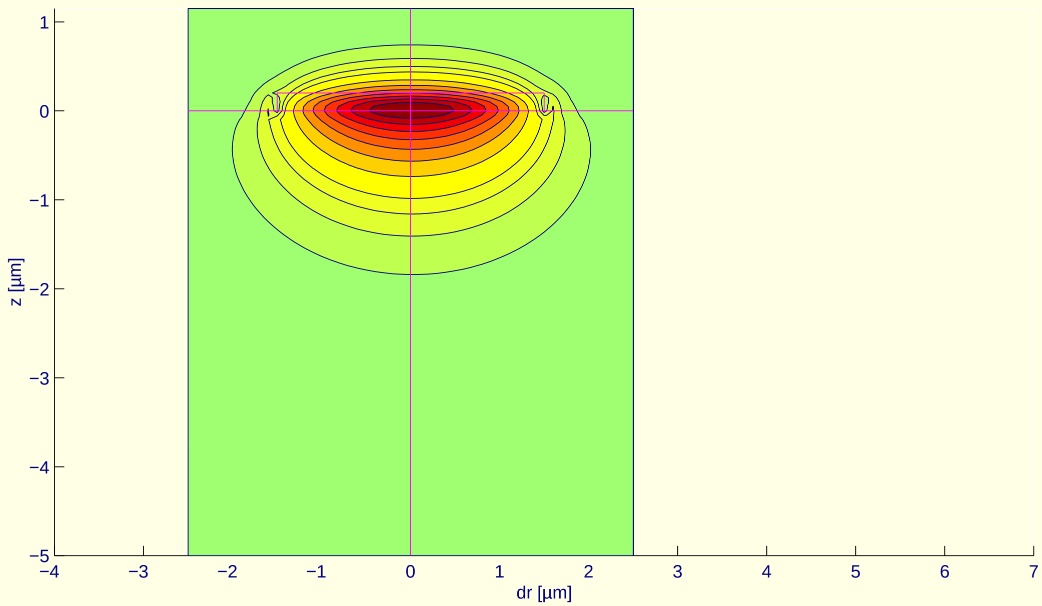
	$w = 3 \div 5\mu m$
$n_1 = 1.989$	$t = 0.1 \div 0.3\mu m$
$n_2 = 1.456$	$h_2 = 5\mu m$
$n_3 = 3.5$	

Sensitivity



Sensitivity depending on thickness t for $w = 3.0\ \mu$m

Distribution of the Radial Electric Field



Fundamental mode, $w = 3.0\ \mu\mathrm{m}$

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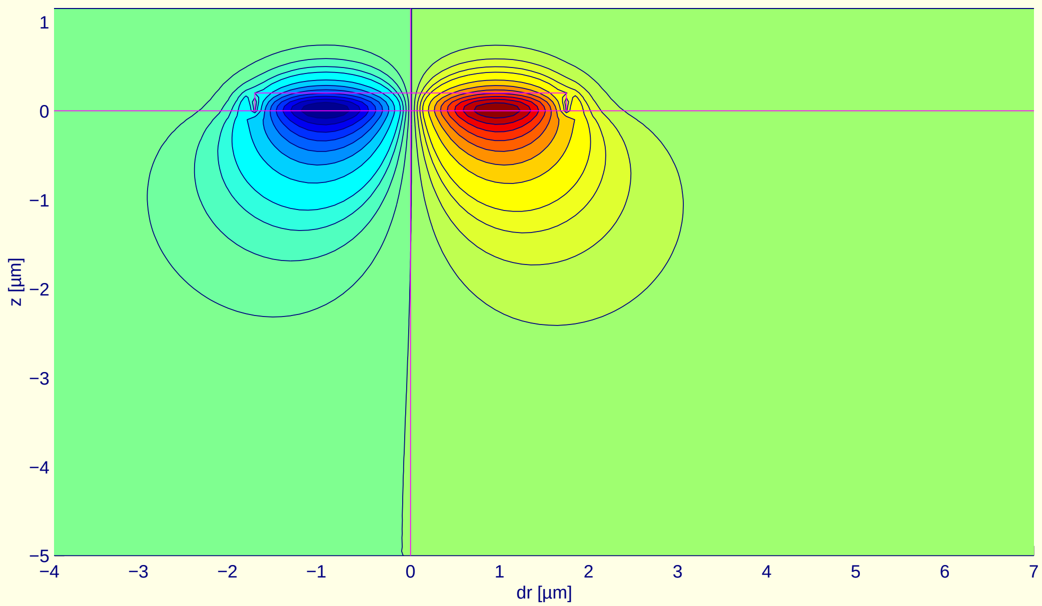

First higher order mode, $w = 3.50\ \mu$m

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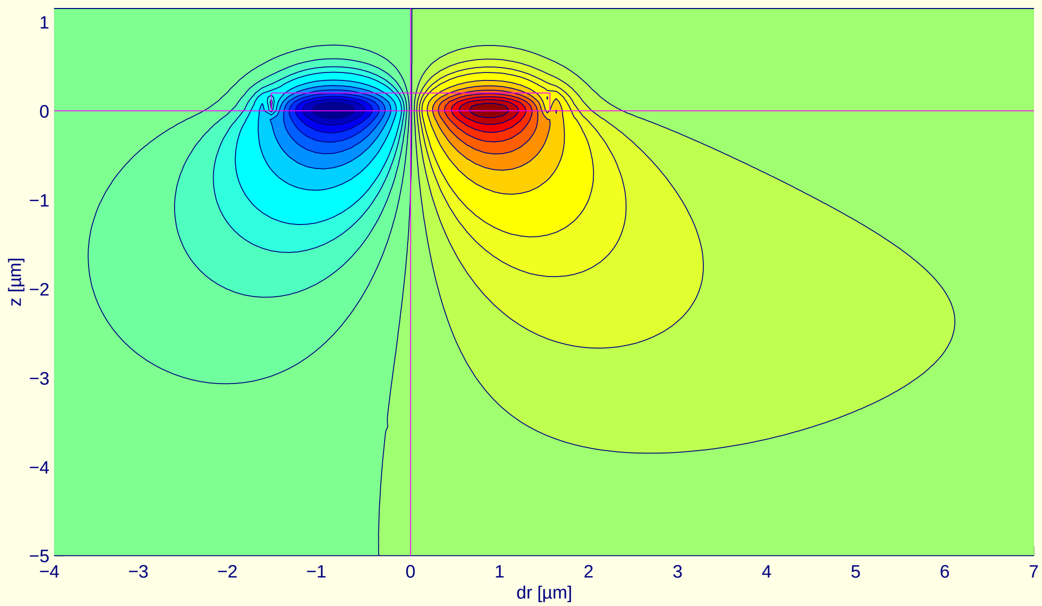

First higher order mode, $w = 3.13\ \mu$m

Conclusion: Advantages and Disadvantages

The presented model yields most accurate results for

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MoL is superior to the Effective Index Method (EIM) and Finite Element Method (FEM)

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Disadvantage

- only applicable to strictly rotational structures

⇒ Extension: Combination with other numerical methods