Pappus' Theorem for a Conic and Mystic Hexagons

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Pappus' Theorem is a well-known result for triples of points on two lines in the (projective) plane:

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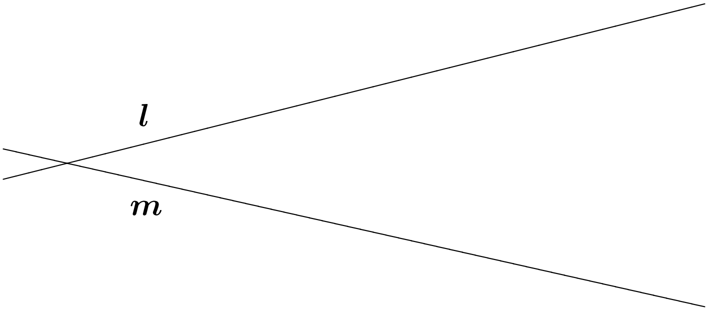
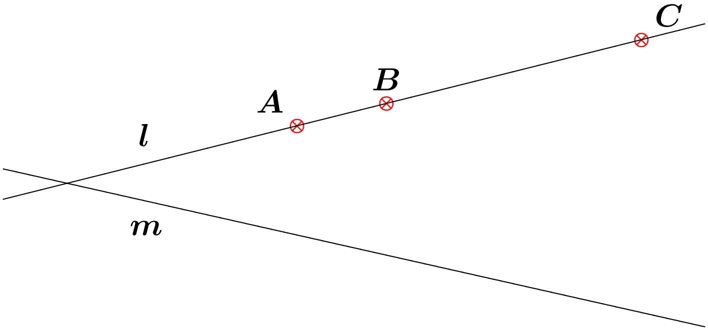
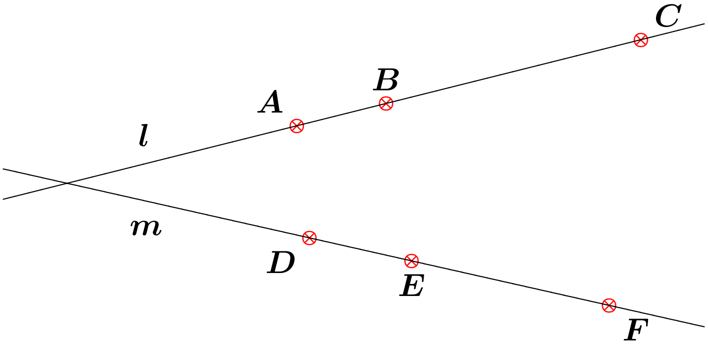


Theorem. (Pappus) *Given lines $\boldsymbol{l}$ and $\boldsymbol{m}$ in the plane,*
three distinct points $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ on $\boldsymbol{l}$ (but not on $\boldsymbol{m}$),



Theorem. (Pappus) *Given lines $\boldsymbol{l}$ and $\boldsymbol{m}$ in the plane, three distinct points $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ on $\boldsymbol{l}$ (but not on $\boldsymbol{m}$), and three distinct points $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{F}$ on $\boldsymbol{m}$ (but not on $\boldsymbol{l}$),*



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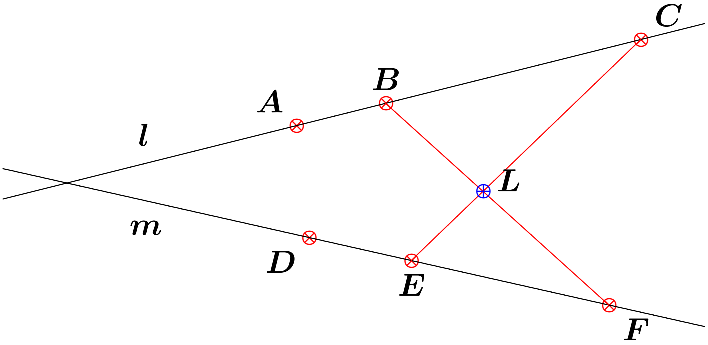
Theorem. (Pappus) *Given lines l and m in the plane, three distinct points A, B, C on l (but not on m), and three distinct points D, E, F on m (but not on l), construct the intersections $L = \overline{BF} \wedge \overline{CE}$,*



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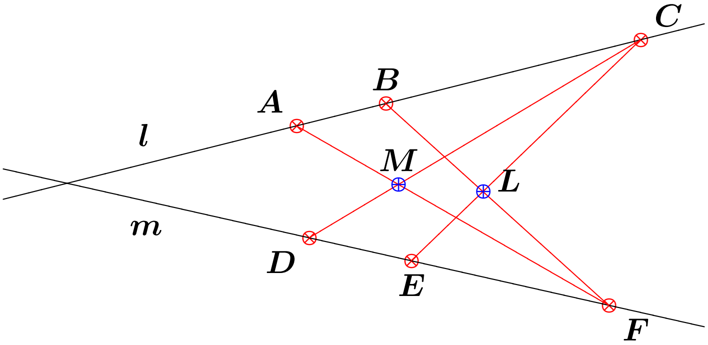

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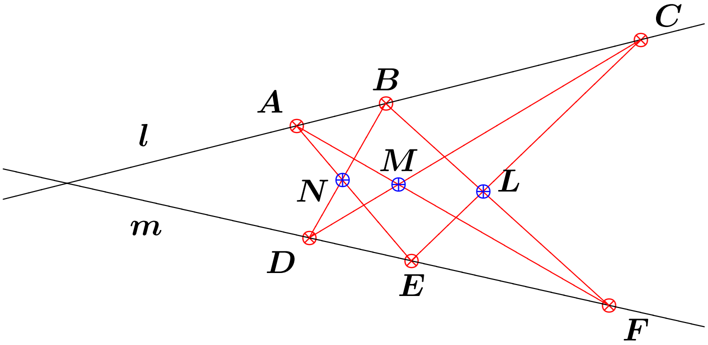
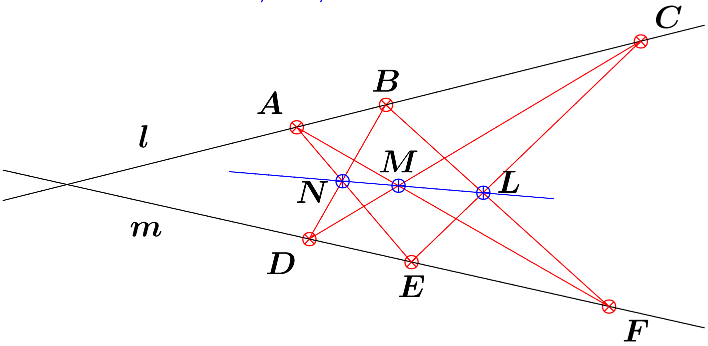
Theorem. (Pappus) *Given lines* $\boldsymbol{l}$ *and* $\boldsymbol{m}$ *in the plane, three distinct points* $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ *on* $\boldsymbol{l}$ *(but not on* $\boldsymbol{m}$*), and three distinct points* $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{F}$ *on* $\boldsymbol{m}$ *(but not on* $\boldsymbol{l}$*), construct the intersections* $\boldsymbol{L} = \overline{\boldsymbol{BF}} \wedge \overline{\boldsymbol{CE}}$, $\boldsymbol{M} = \overline{\boldsymbol{CD}} \wedge \overline{\boldsymbol{AF}}$ *and* $\boldsymbol{N} = \overline{\boldsymbol{AE}} \wedge \overline{\boldsymbol{BD}}$.



Theorem. (Pappus) *Given lines $\boldsymbol{l}$ and $\boldsymbol{m}$ in the plane, three distinct points $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ on $\boldsymbol{l}$ (but not on $\boldsymbol{m}$), and three distinct points $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{F}$ on $\boldsymbol{m}$ (but not on $\boldsymbol{l}$), construct the intersections $\boldsymbol{L} = \overline{\boldsymbol{BF}} \wedge \overline{\boldsymbol{CE}}$, $\boldsymbol{M} = \overline{\boldsymbol{CD}} \wedge \overline{\boldsymbol{AF}}$ and $\boldsymbol{N} = \overline{\boldsymbol{AE}} \wedge \overline{\boldsymbol{BD}}$. Then $\boldsymbol{L}$, $\boldsymbol{M}$, $\boldsymbol{N}$ are collinear.*



Definition. *By a* **Pappus configuration** *we mean a set of 6 points in the plane, arranged as a pair of triples of points (not necessarily collinear)* (A, B, C) *and* (D, E, F), *such that the intersections* (L, M, N), *of pairs of lines joining points taken as in Pappus' Theorem, are collinear.*

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Not so well-known is that if the six points $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{F}$ lie on a non-singular conic, then $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$, $(\boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F})$ form a Pappus configuration.

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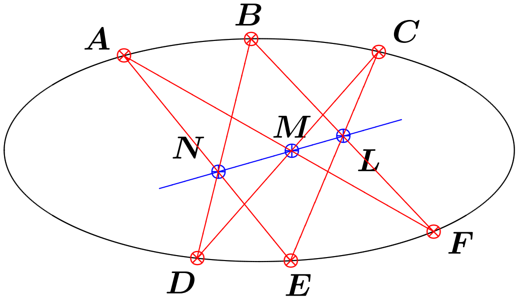

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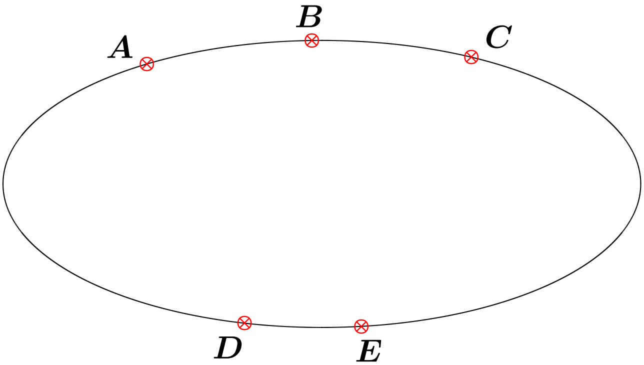
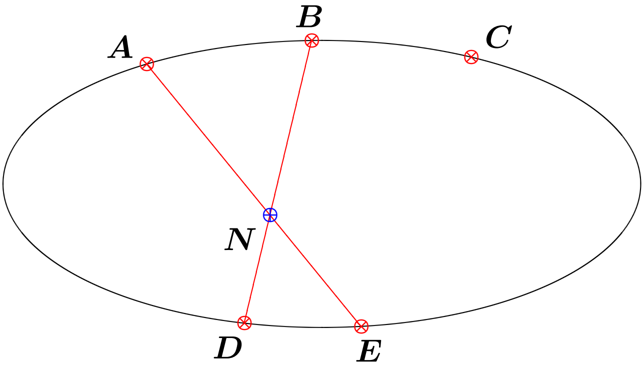
Remark. *It is well known that a unique conic can be drawn through any 5 points in general position.*

Pappus configurations give a way to construct that conic, parametrised by the points on a line...

Given five points A, B, C, D, E in the plane in sufficiently general position,



Given five points $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{E}$ in the plane in sufficiently general position, let $\boldsymbol{N} = \overline{\boldsymbol{AE}} \wedge \overline{\boldsymbol{BD}}$.



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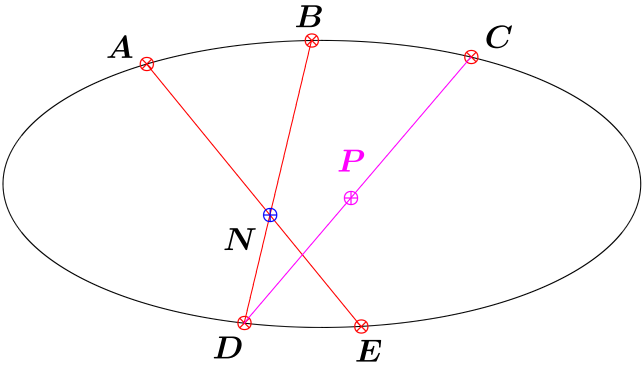

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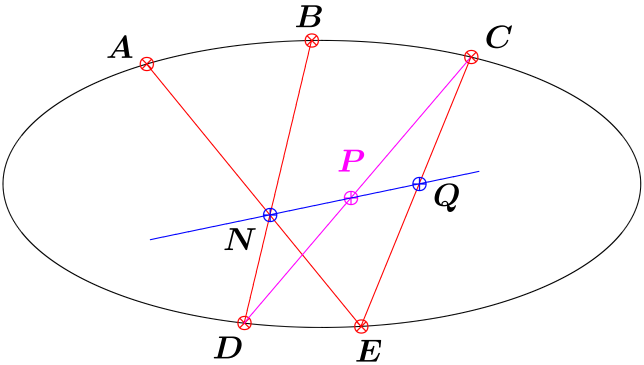

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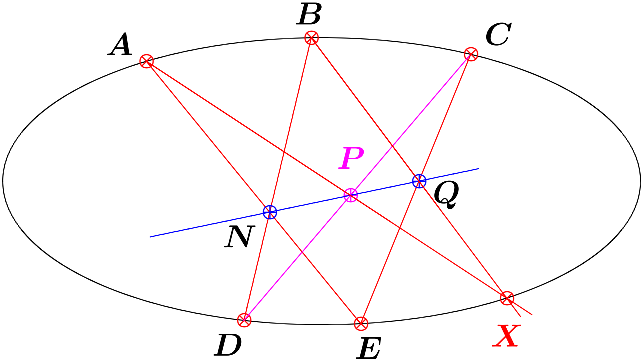
Then $X = \overline{AP} \wedge \overline{BQ}$ lies on the conic.



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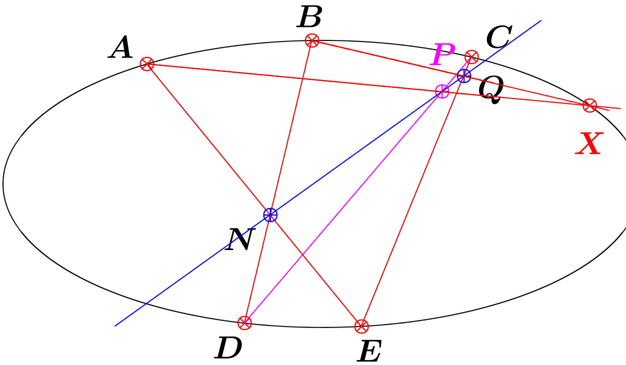
Vary P on $\overline{CD}$ for different points X on the conic:



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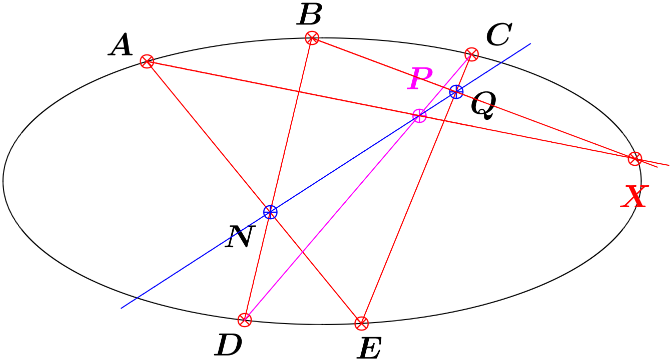

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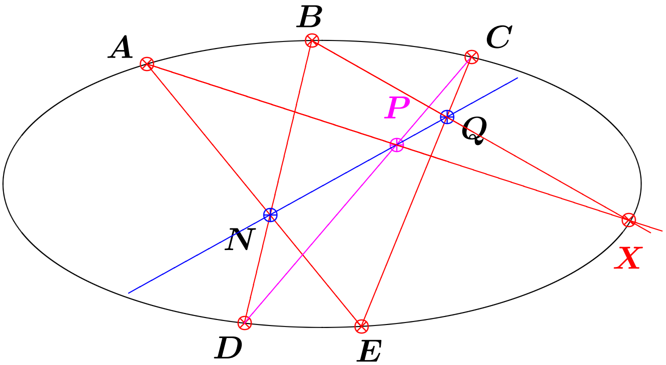

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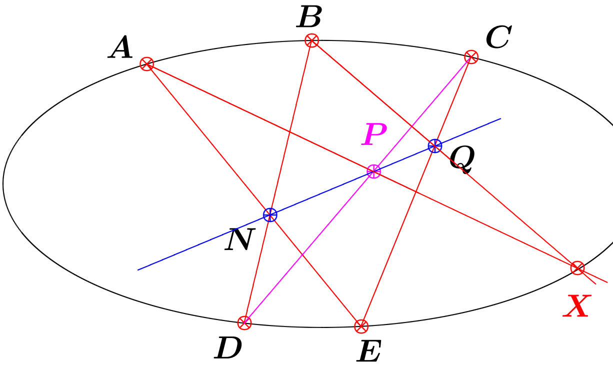

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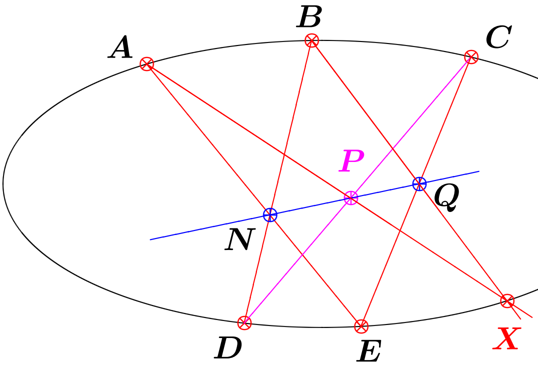

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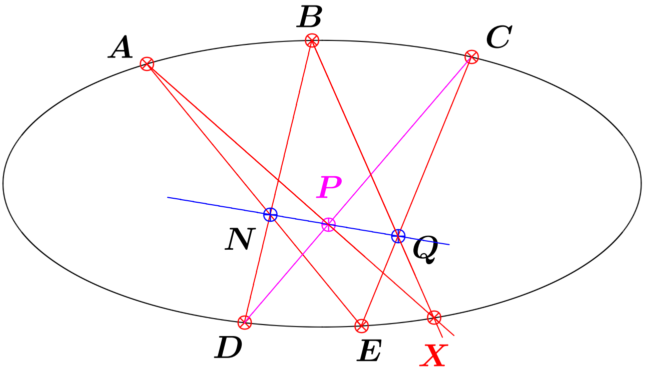

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