

$$\begin{aligned}
c_k &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} \, dx = \frac{1}{2\pi} \sum_{j=1}^r \int_{t_{j-1}}^{t_j} f(x) e^{-ikx} \, dx \\
&= \frac{-i}{2\pi k} \int_0^{2\pi} \varphi(x) e^{-ikx} \, dx = \frac{-i\gamma_k}{k}.
\end{aligned}$$

As for all $\alpha, \beta \in \mathbb{C}$, $|\alpha\beta| \leq \frac{1}{2} (|\alpha|^2 + |\beta|^2)$, it holds that

$$|c_k| \leq \frac{1}{2} \left(\frac{1}{|k|^2} + |\gamma_k|^2 \right).$$

From the convergence of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ and $\sum_{k=-\infty}^{\infty} |\gamma_k|^2$, it follows that

$$\sum_{k=-\infty}^{\infty} |c_k| < \infty.$$