Short Vectors of Planar Lattices via Continued Fractions

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Outline

Lattices

- Definition of planar integral lattices
- Shortest Vectors
- Applications
- The Gaussian algorithm

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Short vectors and best approximations

- Hermite normal form
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- Best approximations and convergents
- A naive shortest vector algorithm based on euclid

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Fast basis reduction

- Identifying the “shortest convergent”
- Computing shortest convergent w.r.t. ℓ_∞-norm with Schönages speedup
- Reduced bases
- Fast basis reduction via (Schönhage 1971) and (Gauß 1801)

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where $A \in \mathbf{Z}^{2\times 2}$ nonsingular

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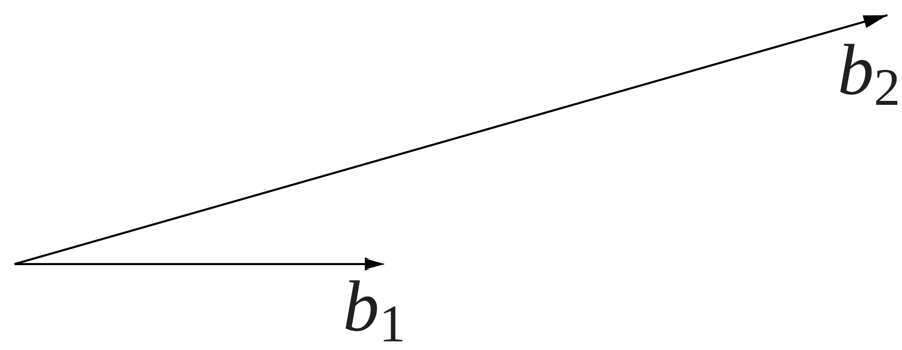

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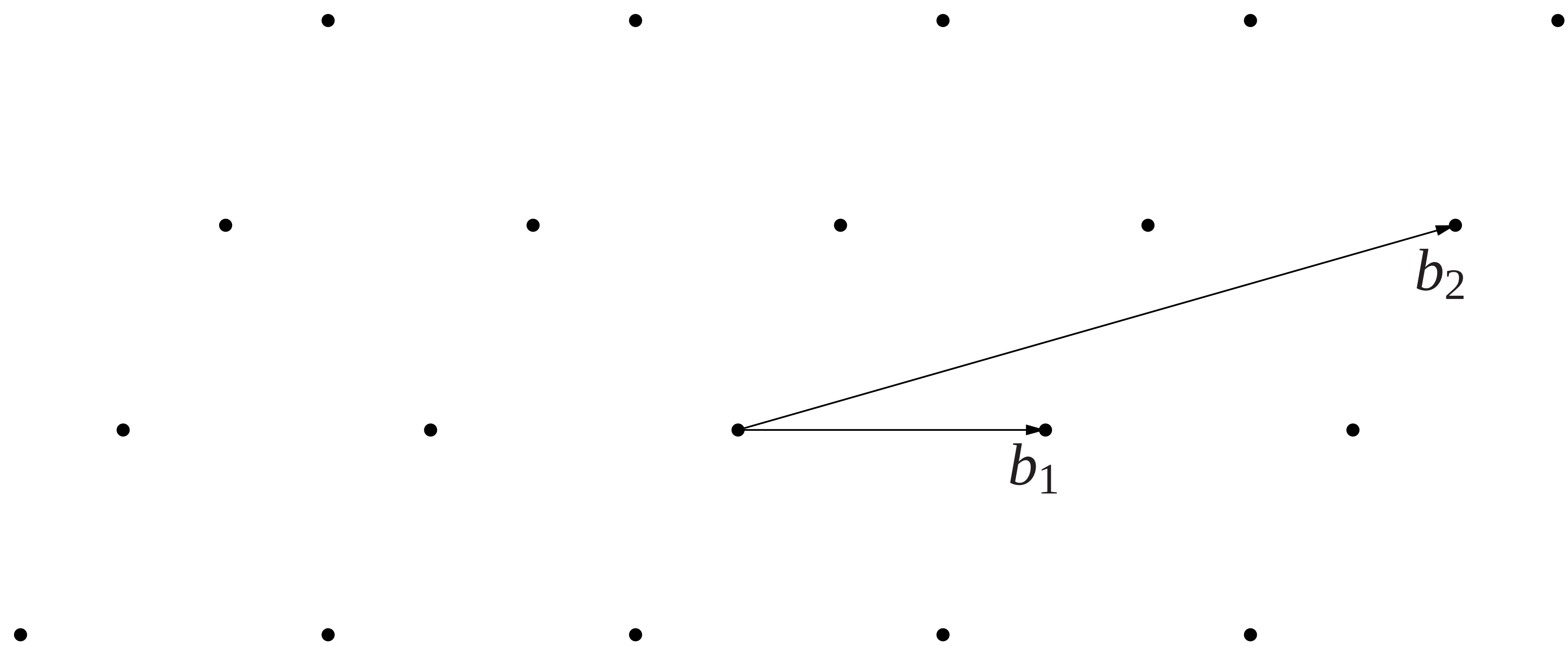

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Examples of unimodular transformations

- swapping of columns
- adding integral multiples of one column to another

The Gaussian reduction algorithm

GAUSS(b_1, b_2)

repeat

arrange that b_1 is the shorter vector of b_1 and b_2

find $k \in \mathbf{Z}$ such that $b_2 - kb_1$ is of minimal euclidean length

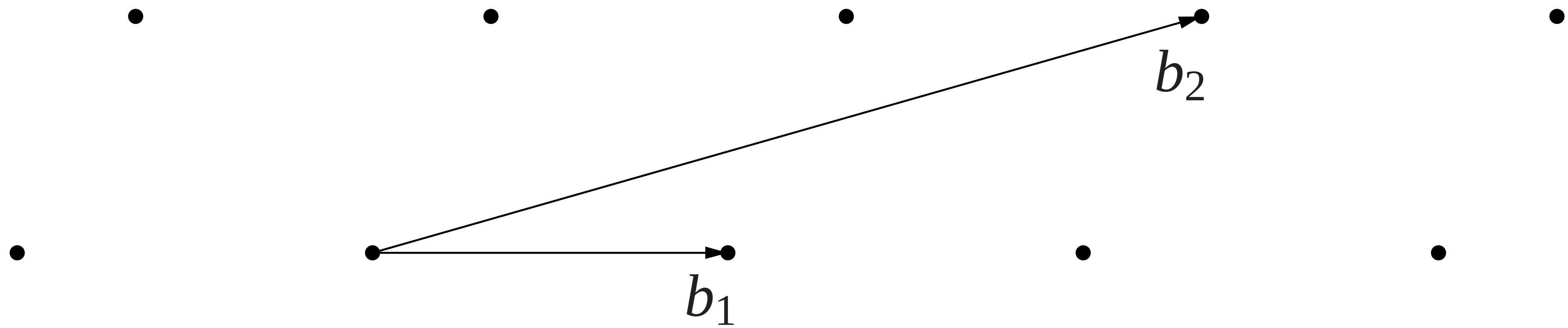
$b_2 \leftarrow (b_2 - kb_1)$ (normalization step)

until $k = 0$

return (b_1, b_2)

k in **repeat**-loop is nearest integer to $(b_1^T b_2)/(b_1^T b_1)$

A normalization step



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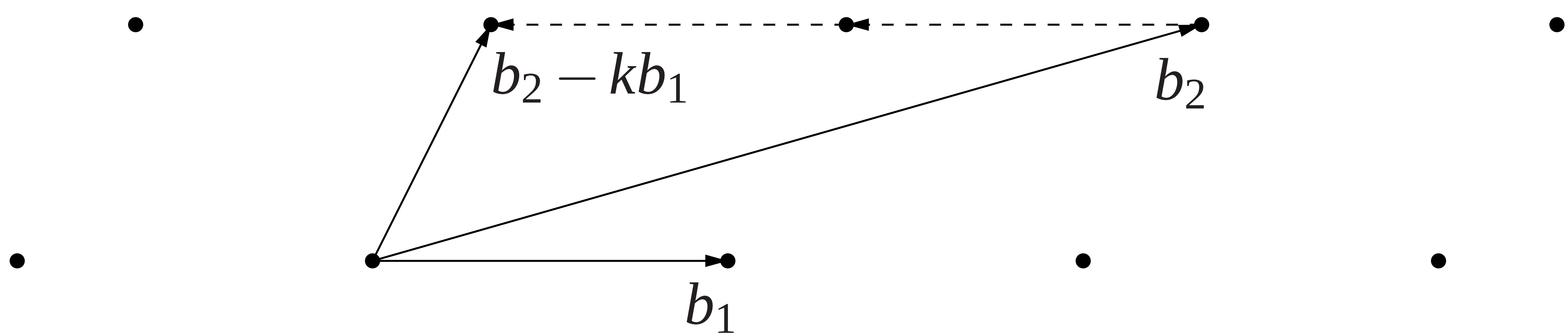

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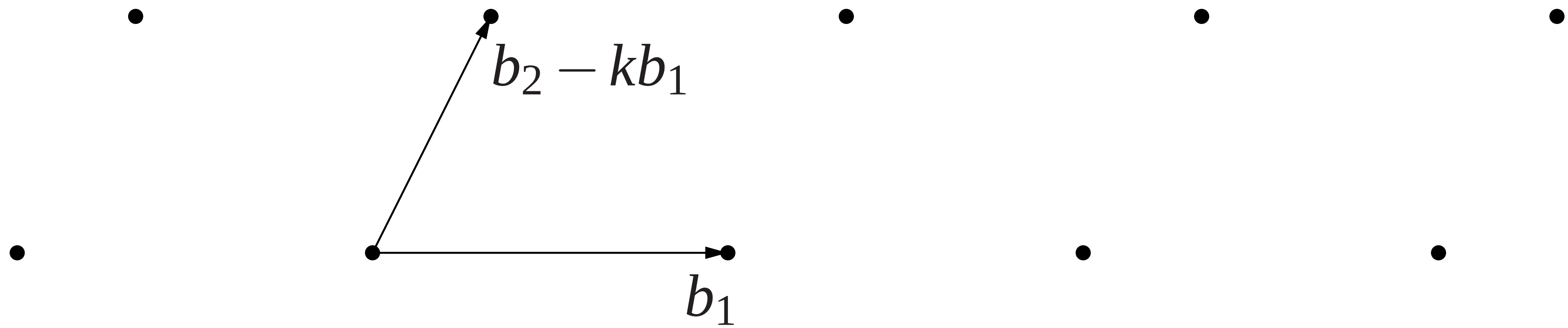

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- This talk: Fast basis reduction via Schönhage's (1971) classical gcd-speedup and GAUSS

The Hermite Normal Form

Given lattice basis $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathbf{Z}^{2\times 2}$, compute integers x, y with $\gcd(a_3, a_4) = d = xa_3 + ya_4$

$\begin{pmatrix} a_4/d & x \\ -a_3/d & y \end{pmatrix}$ unimodular

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We can assume that $c > 0$ and $a > b \geqslant 0$

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Complexity of HNF: $\mathrm{O}(M(n)\log n)$

Assuming: Lattice is given by its Hermite normal form

Best approximations

For $\alpha \in Q$, fraction x/y is best approximation if all other fractions x'/y' with $y' \leqslant y$ satisfy

$$|y'\alpha - x'| > |y\alpha - x|$$

Shortest vectors and best approximations

$$\Lambda = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; x, y \in \mathbf{Z} \right\}$$

Theorem. *There exists a shortest vector* $\binom{-xa+yb}{yc}$, $x \in \mathbf{N}_0$, $y \in \mathbf{N}_+$ *of* Λ *such that at least one of the following conditions is satisfied.*

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- *The fraction* x/y *is a best approximation to the number* b/a.
- *If the fraction* p/q *is the reduced representation of* b/a*, then* p *is odd,* q *is even,* $x \in \{\lfloor p/2 \rfloor, \lceil p/2 \rceil\}$ *and* $y = q/2$.

Best approximations via euclidean algorithm

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- $\lfloor \alpha \rfloor$
- $\lfloor \alpha \rfloor + 1/c$, where c convergent of $1/(\alpha - \lfloor \alpha \rfloor)$

The Gregorian calendar

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Leap year all 4 years

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The first year the in which calendar is 1 day ahead is 4915

Computing convergents

Euclidean algorithm

$\textsc{ExGCD}(a,b)$

$M \leftarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$n \leftarrow 0$

while $(b \neq 0)$ **do**

$\quad q \leftarrow \lfloor a/b \rfloor$

$\quad M \leftarrow M \begin{pmatrix} q & 1 \\ 1 & 0 \end{pmatrix}$

$\quad (a,b) \leftarrow (b, a-qb)$

$\quad n \leftarrow n+1$

return $(d = a, x = (-1)^n M_{2,2}, y = (-1)^{n+1} M_{1,2})$

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Let $M^{(k)}$, $k \geqslant 0$, denote M after k-th iteration of while-loop

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Well known fact:

k-th convergent of a/b is $M^{(k)}_{1,1}/M^{(k)}_{2,1}$

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  Linear search through all convergents of b/a

Finding shortest vector w.r.t ℓ_∞-norm

Consider set of vectors

$$\left\{ \begin{pmatrix} -g_k a + h_k b \\ h_k c \end{pmatrix} \mid k = 0, \ldots, t \right\}, \tag{1}$$

where $\beta_k = g_k / h_k$, $0 \leqslant k \leqslant t$ are the convergents of b/a.

Proposition. *Shortest vector in* (1) *w.r.t.* ℓ_∞ *is last convergent of* b/a *outside the interval* $[(b-c)/a, (b+c)/a]$ *or first convergent of* b/a *inside* $[(b-c)/a, (b+c)/a]$.

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Proposition. *Let $\beta_k = g_k/h_k$ common convergent of $[(b-c)/a, (b+c)/a]$. Then k-th, $k+1$-st or $k+2$-nd convergent of b/a is shortest vector in (1) w.r.t. the ℓ_∞-norm.*

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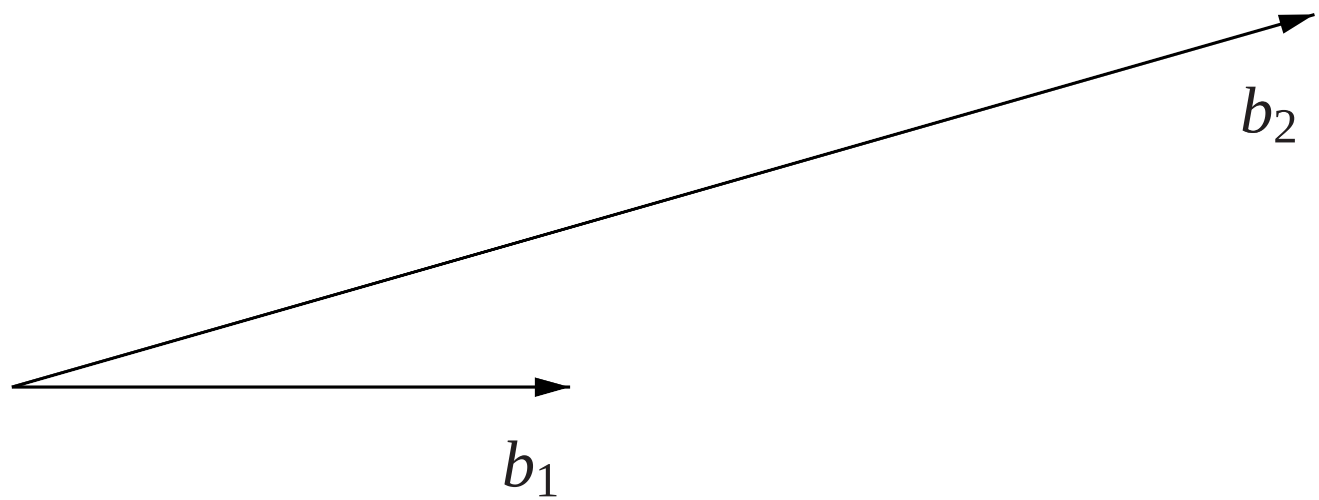
- Compute common convergent β_k of $[(b-c)/a, (b+c)/a]$ and corresponding matrix $M^{(k)}$. Compute next two convergents of b/a with EXGCD; Replace MIN if one of convergents yields shorter vector

Reduced bases

Lattice basis (b_1, b_2) is reduced if enclosed angle in range $90^\circ \pm 30^\circ$



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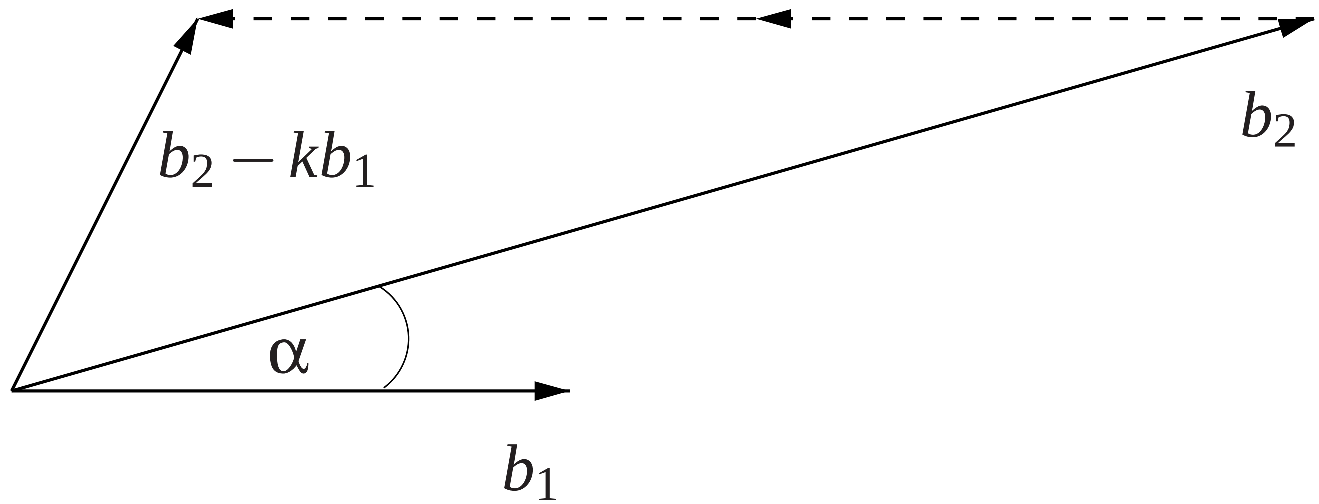

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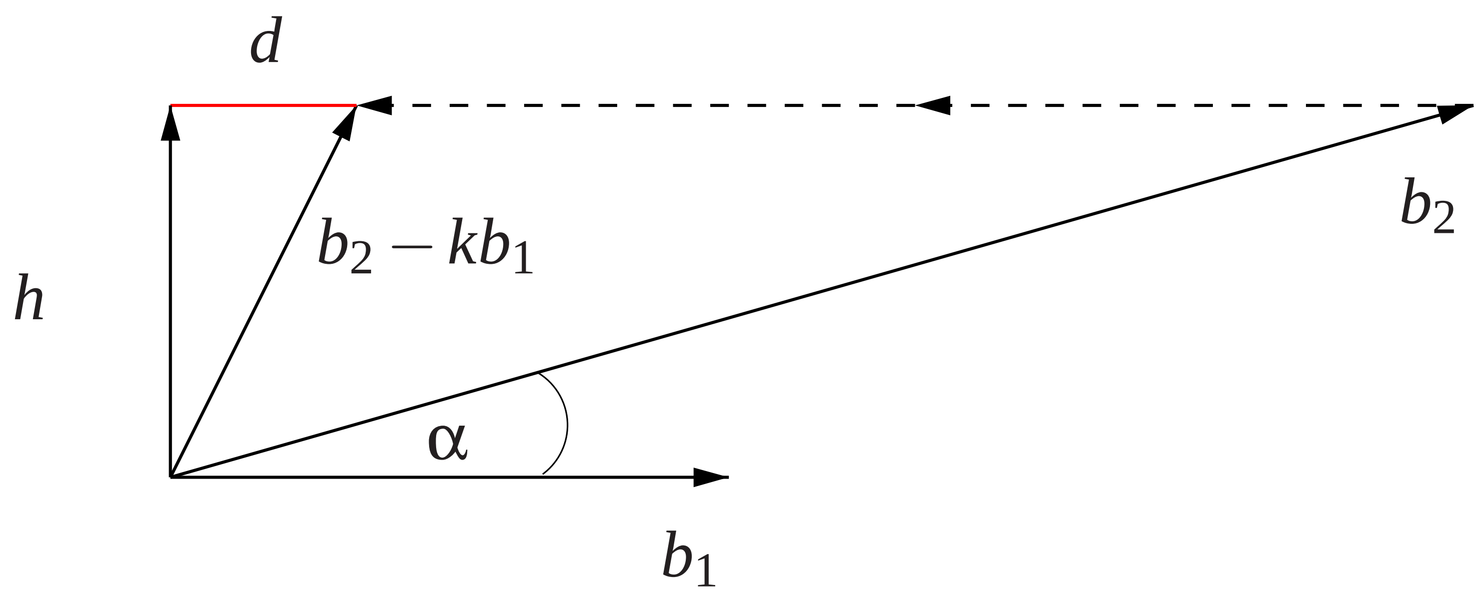

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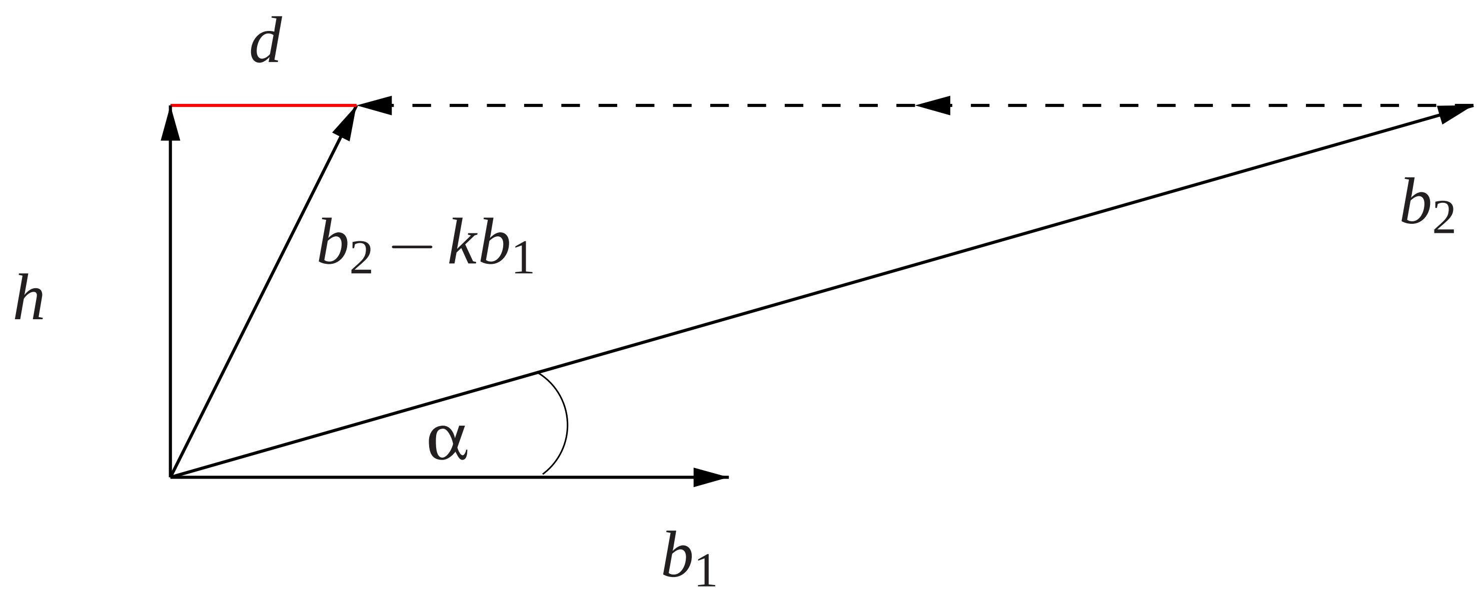

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$$\begin{aligned}\|b_2 - kb_1\|^2 &= \|h\|^2 + \|d\|^2 = 1/4\|b_1\|^2 + (\sin\alpha)^2\|b_2\|^2 \\ &\leqslant (1/4 + (\sin\alpha)^2)\,\|b_2\|^2\end{aligned}$$

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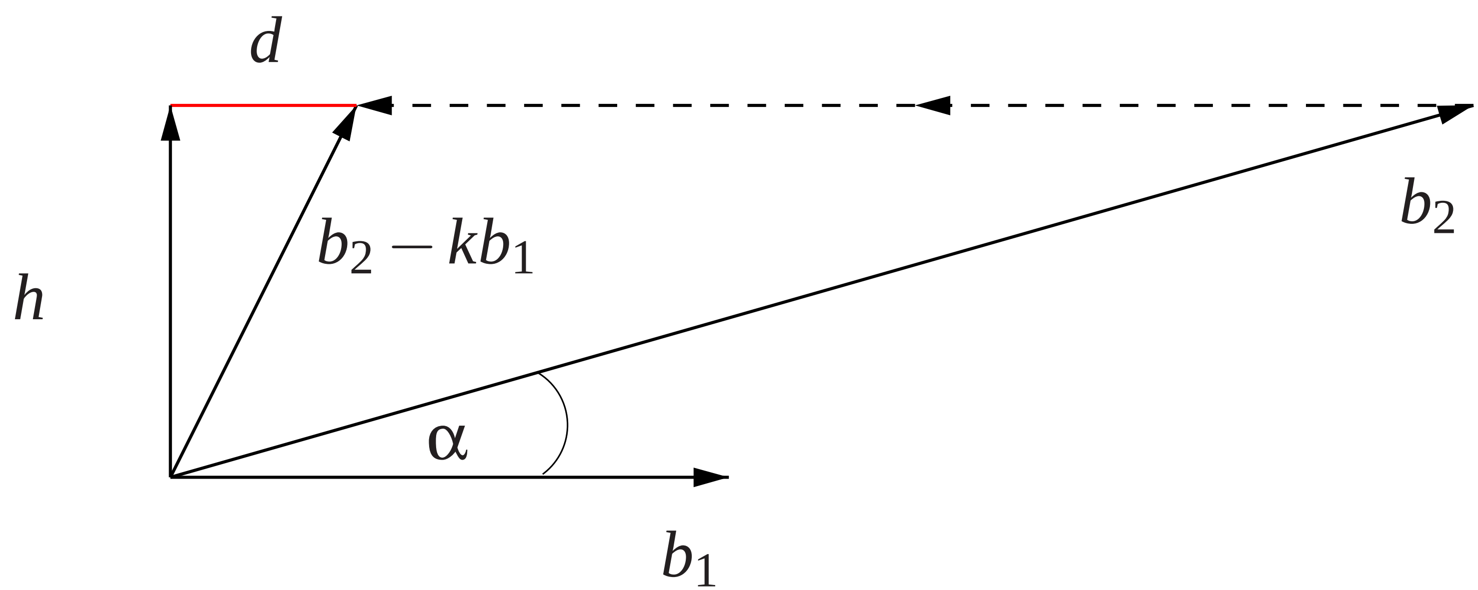

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Almost reduced basis

Proposition. *There exists* $\mathrm{O}(M(n)\log n)$ *time algorithm that computes basis* B *of* Λ *defined by* $A \in \mathbf{Z}^{2\times 2}$*, with property that the first column of* B *is shortest vector w.r.t.* ℓ_∞*-norm.*

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Corollary. *There exists* $\mathrm{O}(M(n)\log n)$ *time algorithm that computes reduced basis* B *of* Λ *defined by* $A \in \mathbf{Z}^{2\times 2}$

Summary of results

- Shortest vector corresponds to best approximation of a rational defined by the lattice
- Shortest vectors can be found with the euclidean algorithm
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Challenge:

Prove that shortest vector problem in arbitrary fixed dimension can be solved in time $O(M(n)\log n)$